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ГРАВИТАЦИОННОЕ ВЗАИМОДЕЙСТВИЕ ДВУХ КОМПЛАНАРНЫХ КОНЦЕНТРИЧЕСКИХ КОЛЕЦ

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GRAVITATIONAL INTERACTION OF TWO COPLANAR CONCENTRIC RINGS

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Аннотация. Рассматривается случай распределения массы по двум компланарным концентрическим кольцам без наличия центрального тела и с его наличием. Получены выражения, позволяющие провести оценку величины напряженности гравитационного поля в точках, локализованных на самих кольцах. Проведена оценка возможных орбитальных скоростей в кольцах.

Ключевые слова: *напряженность гравитационного поля*, *радиус кольца*, *масса кольца.*

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Abstract. The case of mass distribution across two coplanar concentric rings, with and without a central body, is considered. Expressions are derived for estimating the magnitude of the gravitational field at the points located on the rings themselves. Possible orbital velocities within the rings are estimated.

Keywords: *gravitational field strength*, *ring radius*, *ring mass.*

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Introduction

Planetary ring systems are among the most fascinating and mysterious phenomena in astronomy. For a long time, possessing rings was considered a unique feature of Saturn. Saturn was the first planet found to have rings. The total mass of the debris within the ring system is estimated to exceed 10^{19} kilograms.

However, the development of observational technology and the missions of the Voyager interplanetary probes in the second half of the 20th century revolutionized our understanding. Observations revealed that all the giant planets in the Solar System – without exception – are surrounded by rings.

Images from the Voyager-1 interplanetary probe revealed that Saturn's rings consist of hundreds of narrow ringlets. Subsequent images obtained by probes showed that the rings are actually composed of thousands of rings alternating with gaps.

Planetary rings can be described not merely as beautiful adornments, but as dynamic laboratories that offer insight into the processes of celestial body formation. By studying the structure and composition of these systems, one can infer what the Solar System looked like billions of years ago, when modern worlds were beginning to form from the protoplanetary disk. It is worth noting that the recent discovery of rings around small bodies such as the asteroid Chariklo and the dwarf planet Haumea demonstrates that such structures are far more common in the Universe than previously thought.

Saturn's rings have been studied by P. Laplace, S. V. Kovalevskaya, J. Maxwell, and M. S. Bobrov [1]–[4]. The gravitational interaction between two coplanar concentric rings is examined below. A two-ring system has actually been observed around the asteroid Chariklo. It is one of the largest Centaurs and the largest asteroid between the Main Belt and the Kuiper Belt discovered in 1997 (Figure 0.1). The existence of a ring system around this asteroid was announced in 2014 (following an occultation of a star by the Centaur in 2013).

An international team of astronomers led by Felipe Braga-Ribas, using seven telescopes in South America, detected not only the primary drop in the star's brightness as it passed behind the asteroid's disk but also two symmetrical secondary drops before and after the main event [5]. This indicated the presence of dense, narrow structures in orbit around the small body.

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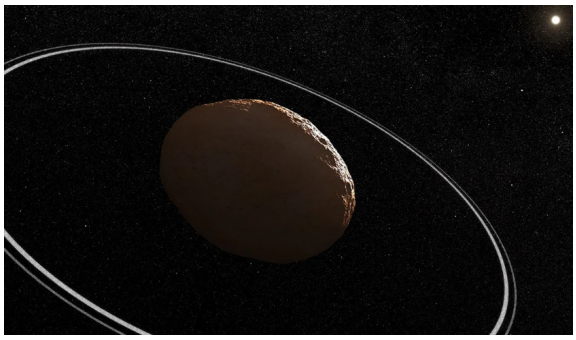


Figure 0.1 – Rings of the asteroid Chariklo (source: NASA)

In the model under consideration, we assume the asteroid to be a homogeneous sphere and the rings to be circular.

1 Determination of gravitational field strength

Gravitational field strength of a ring for points in the plane. According to works [6], [7], the gravitational field strength of a ring at points in the plane in which the ring lies can be expressed as

$$g = -\frac{Gm}{\pi R^2}\int_0^{\pi}\frac{(k-\cos\alpha)d\alpha}{\left(1+k^2-2k\cos\alpha\right)^{1,5}}, \qquad (1.1)$$

where G – is the gravitational constant, m – is the mass of the ring, R – is the radius of the ring, $k = r / R$, and r – is the distance from the center of the ring to the point under consideration.

For points located inside the ring, the parameter $k < 1$; for points located outside the ring, $k > 1$. At $k = 1$, a logarithmic divergence occurs due to the model's idealization, in which the volumetric mass density tends to infinity (the ring's cross-section is assumed to be zero, and the mass is distributed along the circumference). In [7], it is proposed to use a specific expression to estimate the average gravitational field strength at points belonging to the ring.

$$g = -\frac{5Gm}{2\pi R^2}. \qquad (1.2)$$

Gravitational field strength of a two-ring system at the points of their location. Let us consider a system of two rings with radii R_1 and R_2 and masses m_1 and m_2, respectively (Figure 1.1).

The gravitational field strength of the first ring (of radius R_1) at the points where the second ring (of radius R_2) is located is expressed, according to (1.1), as

$$g_{12} = -\frac{Gm_1}{2\pi R_1^2}\int_0^{\pi}\frac{\frac{R_2}{R_1}-\cos\alpha}{\left(1+\left(\frac{R_2}{R_1}\right)^2-2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}}d\alpha. \qquad (1.3)$$

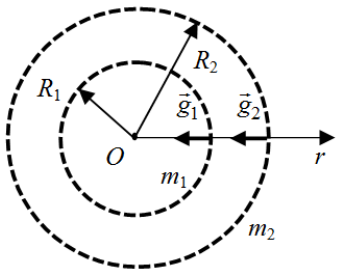


Figure 1.1 – Inner ring of radius R_1 and mass m_1, outer ring of radius R_2 and mass m_2

The gravitational field strength of the second ring (of radius R_2) at the locations of the first ring (of radius R_1) is expressed, according to (1.1), as follows:

$$g_{21} = -\frac{Gm_2}{2\pi R_2^2}\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha. \qquad (1.4)$$

The gravitational field strength of both rings at the points of localization of the first ring is expressed, according to (1.2) and (1.4), as

$$g_1 = -\frac{Gm_2}{2\pi R_2^2}\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha - \frac{5Gm_1}{2\pi R_1^2}. \qquad (1.5)$$

The gravitational field strength of both rings at the points of localization of the second ring is expressed, according to (1.2) and (1.3), as follows:

$$g_2 = -\frac{Gm_1}{2\pi R_1^2}\int_0^{\pi}\frac{\frac{R_2}{R_1}-\cos\alpha}{\left(1+\left(\frac{R_2}{R_1}\right)^2-2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}}d\alpha - \frac{5Gm_2}{2\pi R_2^2}. \qquad (1.6)$$

Gravitational field strength of the central body and the two-ring system at the points where the rings are located. The gravitational field strength of a central body of mass M at the locations of the first ring of radius R_1 and the second ring of radius R_2 (Figure 1.2) is given by expressions (1.7) and (1.8), respectively.

$$g_{01} = -\frac{GM}{R_1^2}, \qquad (1.7)$$

$$g_{02} = -\frac{GM}{R_2^2}. \qquad (1.8)$$

Gravitational field strength of the central body and the two-ring system at the locations of the first ring, taking into account (1.5) and (1.7)

$$g_3 = -\frac{Gm_2}{2\pi R_2^2}\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha - \\ -\frac{5Gm_1}{2\pi R_1^2}-\frac{GM}{R_1^2}. \qquad (1.9)$$

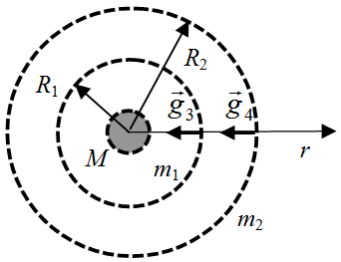


Figure 1.2 – Central body of mass M, inner ring of radius R_1 and mass m_1, outer ring of radius R_2 and mass m_2

Gravitational field strength of the central body and the two-ring system at the locations of the second ring, taking into account (1.6) and (1.8)

$$g_4 = -\frac{Gm_1}{2\pi R_1^2}\int_0^{\pi}\frac{\frac{R_2}{R_1}-\cos\alpha}{\left(1+\left(\frac{R_2}{R_1}\right)^2-2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}}d\alpha - \\ -\frac{5Gm_2}{2\pi R_2^2}-\frac{GM}{R_2^2}. \qquad (1.10)$$

2 Estimation of orbital velocities in the rings

Estimation of orbital velocities in the rings without accounting for the central body. The magnitude of the gravitational field intensity produced by both rings at the points where the first ring is located must equal the centripetal acceleration of the points on the first ring, that is,

$$\frac{Gm_2}{2\pi R_2^2}\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha + \\ +\frac{5Gm_1}{2\pi R_1^2}=\frac{v_1^2}{R_1}. \qquad (2.1)$$

The gravitational field strength of both rings at the points where the second ring is located must equal, in magnitude, the centripetal acceleration of the points on the second ring, that is,

$$\frac{Gm_1}{2\pi R_1^2}\int_0^{\pi}\frac{\frac{R_2}{R_1}-\cos\alpha}{\left(1+\left(\frac{R_2}{R_1}\right)^2-2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}}d\alpha + \\ +\frac{5Gm_2}{2\pi R_2^2}=\frac{v_2^2}{R_2}. \qquad (2.2)$$

The velocity of points on the first ring is determined from relation (2.1).

$$v_1 = \sqrt{\frac{Gm_2}{2\pi R_2^2}R_1\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha+\frac{5Gm_1}{2\pi R_1}}. \qquad (2.3)$$

The velocity of points on the second ring is determined from relation (2.2).

$$v_2 = \sqrt{\frac{Gm_1}{2\pi R_1^2}R_2\int_0^{\pi}\frac{\frac{R_2}{R_1}-\cos\alpha}{\left(1+\left(\frac{R_2}{R_1}\right)^2-2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}}d\alpha+\frac{5Gm_2}{2\pi R_2}}. \qquad (2.4)$$

Substituting the following values for the radii and masses of the rings (data taken from [8]) into relations (2.3) and (2.4) – $R_1 = 390.6$ km, $R_2 = 404.8$ km, $m_1 = 9.8 \cdot 10^{15}$ kg ($9.8 \cdot 10^{18}$ g), $m_2 = 10^{15}$ kg (10^{18} g) – we obtain $v_1 = 0.81$ m/s and $v_2 = 2.86$ m/s. These results indicate that such a scenario is theoretically possible (even in the absence of a central body) and that there is a significant difference in the speeds of the rings despite their proximity. Notably, the speed of the inner ring is lower than that of the outer ring.

Estimation of orbital velocities in rings, accounting for the central body. The gravitational field strength of the central body and the two-ring system at the locations of the first ring must equal, in magnitude, the centripetal acceleration of points on the first ring, that is,

$$\frac{Gm_2}{2\pi R_2^2}\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha + \\ +\frac{5Gm_1}{2\pi R_1^2}+\frac{GM}{R_1^2}=\frac{v_3^2}{R_1}. \qquad (2.5)$$

The gravitational field strength of the central body and the system of two rings at the locations of the second ring must equal the magnitude of the centripetal acceleration of points on the second ring, that is,

$$\frac{Gm_1}{2\pi R_1^2}\int_0^{\pi}\frac{\frac{R_2}{R_1}-\cos\alpha}{\left(1+\left(\frac{R_2}{R_1}\right)^2-2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}}d\alpha + \\ +\frac{5Gm_2}{2\pi R_2^2}+\frac{GM}{R_2^2}=\frac{v_4^2}{R_2}. \qquad (2.6)$$

The velocity of points on the first ring is determined from relation (2.5).

$$v_3 = \left(\frac{Gm_2}{2\pi R_2^2}R_1\int_0^{\pi}\frac{\frac{R_1}{R_2}-\cos\alpha}{\left(1+\left(\frac{R_1}{R_2}\right)^2-2\frac{R_1}{R_2}\cos\alpha\right)^{1,5}}d\alpha + \\ +\frac{5Gm_1}{2\pi R_1}+\frac{GM}{R_1}\right)^{\frac{1}{2}}. \qquad (2.7)$$

The velocity of points on the second ring is determined from relation (2.6).

$$v_4 = \left(\frac{Gm_1}{2\pi R_1^2} R_2 \int_0^{\pi} \frac{\frac{R_2}{R_1} - \cos\alpha}{\left(1 + \left(\frac{R_2}{R_1}\right)^2 - 2\frac{R_2}{R_1}\cos\alpha\right)^{1,5}} d\alpha + \right.$$

$$\left. + \frac{5Gm_2}{2\pi R_2} + \frac{GM}{R_2} \right)^{\frac{1}{2}}. \qquad (2.8)$$

Let us substitute the following values into relations (2.7) and (2.8): $R_1 = 390.6$ km, $R_2 = 404.8$ km, $m_1 = 9.8 \cdot 10^{15}$ kg, $m_2 = 10^{15}$ kg, $M = 7 \cdot 10^{18}$ kg (the estimated mass of the asteroid is $(6–8) \cdot 10^{18}$ kg).

Upon calculation, we obtain $v_3 = 34.58$ m/s and $v_4 = 34.08$ m/s.

The Keplerian velocities for points on the first ring will be

$$v_1^* = \sqrt{\frac{GM}{R_1}} = 34,57 \text{ m/s}. \qquad (2.9)$$

The Keplerian velocities for points on the second ring will amount to a value of

$$v_2^* = \sqrt{\frac{GM}{R_2}} = 33,96 \text{ m/s}. \qquad (2.10)$$

The difference in particle velocities in the rings, disregarding the gravitational influence of the rings (expressions (2.9) and (2.10)) $v_1^* - v_2^* = 0,61$ m/s.

The difference in particle velocities in the rings, accounting for the gravitational influence of the rings $v_3 - v_4 = 0,5$ m/s, amounts to 82% of $v_1^* - v_2^*$ the difference in Keplerian velocities, which is 18% less.

Conclusion

Thus, by examining cases of mass distribution across two coplanar concentric rings both with and without a central body the expressions were derived to estimate the gravitational field strength at points located on the rings themselves. This enabled an assessment of potential orbital velocities within the rings. The calculations performed for the case without a central body revealed the possibility of the outer ring's velocity exceeding that of the inner ring. The calculations involving a central body showed that, when the gravitational influence of the rings is taken into account, the velocity difference between the rings is smaller than that predicted by a Keplerian velocity distribution.

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