

О $\sigma_{\mathfrak{Z}}$ -СВОЙСТВАХ КОНЕЧНЫХ ГРУПП I

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Аннотация. В данной работе все группы конечны, G всегда обозначает конечную группу. $\sigma_{\mathfrak{Z}}$ -Парой называется всякая пара $\sigma, \mathfrak{Z}$, где $\sigma = \{\sigma_i \mid i \in I = \{0\} \cup I\}$ – некоторое разбиение множества всех простых чисел и $\mathfrak{Z}$ – полный (в смысле Виландта) класс σ_0 -групп, содержащий все разрешимые σ_0 -группы. Группа G называется: (i) $\sigma_{\mathfrak{Z}}$ -примарной, если G является σ_i -группой для некоторого $i \in I$ и $G \in \mathfrak{Z}$, если $i = 0$; (ii) $\sigma_{\mathfrak{Z}}$ -нильпотентной, если каждый главный фактор H/K группы G является $\sigma_{\mathfrak{Z}}$ -центральным в G , т.е. полупрямое произведение $(H/K) \times (G/C_G(H/K))$ является $\sigma_{\mathfrak{Z}}$ -примарным; (iii) $\sigma_{\mathfrak{Z}}$ -разрешимой, если каждый главный фактор группы G является $\sigma_{\mathfrak{Z}}$ -примарным. Под $\sigma_{\mathfrak{Z}}$ -свойством группы мы понимаем любое из ее свойств, зависящее от $\sigma_{\mathfrak{Z}}$ и не подразумевающее никаких ограничений на $\sigma_{\mathfrak{Z}}$. В данной работе мы разрабатываем новый аспект теории $\sigma_{\mathfrak{Z}}$ -свойств, связанный с приложениями $\sigma_{\mathfrak{Z}}$ -пар. В частности, на множестве всех $\sigma_{\mathfrak{Z}}$ -пар мы определяем частичный порядок $\leq$ и доказываем, что если $\Sigma = \{\sigma_{\mathfrak{Z}_j}^j \mid j \in J\}$ – множество всех $\sigma_{\mathfrak{Z}}$ -пар таких, что каждая группа в $\mathfrak{X}$ является $\sigma_{\mathfrak{Z}_j}^j$ -нильпотентной (соответственно, $\sigma_{\mathfrak{Z}_j}^j$ -разрешимой) для всех j , то относительно $\leq$ существует наименьший элемент в Σ .

Ключевые слова: конечная группа, $\sigma_{\mathfrak{Z}}$ -свойство группы, $\sigma_{\mathfrak{Z}}$ -пара, $\sigma_{\mathfrak{Z}}$ -примарная группа, решетка $\sigma_{\mathfrak{Z}}$ -пар.

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Abstract. In this paper all groups are finite, G always denotes a finite group. A $\sigma_{\mathfrak{Z}}$ -pair is any pair $\sigma, \mathfrak{Z}$, where $\sigma = \{\sigma_i \mid i \in I = \{0\} \cup I \subseteq \mathbb{N}\}$ is some partition of the set of all primes and $\mathfrak{Z}$ is a complete (in the sense of Wielandt) class of σ_0 -groups which contains all soluble σ_0 -groups. A group G is said to be: (i) $\sigma_{\mathfrak{Z}}$ -primary provided G is a σ_i -group for some $i \in I$ and $G \in \mathfrak{Z}$ if $i = 0$; (ii) $\sigma_{\mathfrak{Z}}$ -nilpotent if every chief factor H/K of G is $\sigma_{\mathfrak{Z}}$ -central in G , that is, the semidirect $(H/K) \times (G/C_G(H/K))$ is $\sigma_{\mathfrak{Z}}$ -primary; (iii) $\sigma_{\mathfrak{Z}}$ -soluble if every chief factor of G is $\sigma_{\mathfrak{Z}}$ -primary. By a $\sigma_{\mathfrak{Z}}$ -property of a group we mean any of its properties, that depends on $\sigma_{\mathfrak{Z}}$ and which does not imply any restrictions on $\sigma_{\mathfrak{Z}}$. In this paper, we develop a new aspect of the theory of $\sigma_{\mathfrak{Z}}$ -properties related to applications of $\sigma_{\mathfrak{Z}}$ -pairs. In particular, on the set of all $\sigma_{\mathfrak{Z}}$ -pairs we define a partial order $\leq$ and we prove that if $\Sigma = \{\sigma_{\mathfrak{Z}_j}^j \mid j \in J\}$ is the set of all $\sigma_{\mathfrak{Z}}$ -pairs such that every group in $\mathfrak{X}$ is $\sigma_{\mathfrak{Z}_j}^j$ -nilpotent (respectively, $\sigma_{\mathfrak{Z}_j}^j$ -soluble) for all j , then with respect to $\leq$ there exists the least element in Σ .

Keywords: finite group, $\sigma_{\mathfrak{Z}}$ -property of a group, $\sigma_{\mathfrak{Z}}$ -pair, $\sigma_{\mathfrak{Z}}$ -primary group, the lattice of $\sigma_{\mathfrak{Z}}$ -pairs.

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Introduction

Throughout this paper, all groups are finite and G always denotes a finite group; $\pi(G)$ is the set all primes dividing the order $|G|$ of G . We say,

following Wielandt [1], that a class of groups $\mathfrak{F}$ is *complete* provided $\mathfrak{F}$ is closed under taking extensions, epimorphic images and subgroups.

Definition 0.1. (i) A $(\sigma, \mathfrak{S})$ -pair (or, for brevity, a σ_3 -pair) is any pair $\sigma, \mathfrak{S}$, where

$$\sigma = \{\sigma_i \mid i \in I = \{0\} \cup I\}$$

is some partition of the set of all primes $\mathbb{P}$ and $\mathfrak{S}$ is a complete class of σ_0 -groups which contains all soluble σ_0 -groups.

(ii) We will also use the term a σ_3 -pair as a general name for pairs of this kind, and if $\sigma^* = \{\sigma_i^* \mid i \in I\}$ is any partition of $\mathbb{P}$, then we use $\sigma^*(G)$, following [2], [3], to denote the set $\{\sigma_i^* \mid \sigma_i^* \cap \pi(G) \neq \emptyset\}$.

(iii) By a σ_3 -property of a group we mean any of its properties, that depends on σ_3 and which does not imply any restrictions on σ_3 .

In particular, a σ -property of a group is any of its properties, that depends on the partition σ of $\mathbb{P}$ and which does not imply any restrictions on σ [4]–[7].

The study of the σ_3 -properties goes back to the author's reviews [2], [8] and the recent papers [9], [10].

In this paper, we develop a new aspect of the theory of σ_3 -properties related to applications of σ_3 -pairs.

Definition 0.2. We say that a group G is: (i) σ_3 -primary provided G is a σ_i -group for some $i \in I$ and $G \in \mathfrak{S}$ if $i = 0$;

(ii) σ_3 -nilpotent if every chief factor H/K of G is σ_3 -central in G , that is, the semidirect $(H/K) \rtimes (G/C_G(H/K))$ is σ_3 -primary;

(iii) σ_3 -soluble if every chief factor of G is σ_3 -primary.

Definition 0.3. We say that a subgroup A of G is σ_3 -subnormal in G if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ_3 -primary for all $i = 1, \dots, t$.

Remark 0.4. (i) In what follows, we always omit the symbol $\mathfrak{S}$ in all definitions and notations in the case when $\mathfrak{S}$ is the class of all σ_0 -groups (note, in passing, that in view of the Burnside's $p^a q^b$ -theorem and the Feit–Thompson theorem, we always deal with such a situation when either $|\sigma_0| \leq 2$ or $2 \notin \sigma_0$).

Thus we say, in this case, that G is [5]: σ -primary if G is a σ_i -group for some $i \in I$; σ -nilpotent if every chief factor H/K of G is

σ -central in G ; σ -soluble if every chief factor of G is σ -primary.

We say also that a subgroup A of G is σ -subnormal in G [5] if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ -primary for all $i = 1, \dots, n$.

(ii) In view of Part (i), we will always suppose that $2 \in \sigma_0$.

(iii) Let $G \neq 1$ and $\sigma(G) = \{\sigma_{i_1}, \dots, \sigma_{i_t}\}$. Then G is σ_3 -nilpotent if and only if $G = H_1 \times \dots \times H_t$, where H_j is a Hall σ_{i_j} -subgroup of G for all $j = 1, \dots, t$ and $H_k \in \mathfrak{S}$ for $i_k = 0$ (see Theorem 0.19 (ix) and Lemma 1.1 in [9]).

Definition 0.5. We say that: (i) a collection of groups $\mathfrak{X}$ is σ_3 -nilpotent (σ_3 -soluble) provided every group in $\mathfrak{X}$ is σ_3 -nilpotent (respectively, σ_3 -soluble);

(ii) a set of subgroups $\mathfrak{X}$ of G is σ_3 -subnormal in G provided every subgroup $A \in \mathfrak{X}$ is σ_3 -subnormal in G .

Example 0.6. (i) In the limiting case, when $\sigma = \{\mathbb{P}\}$, $\sigma_0 = \mathbb{P}$ and all soluble groups are contained in $\mathfrak{S}$.

A group G is σ_3 -primary (respectively, σ_3 -nilpotent, σ_3 -soluble) if and only if $G \in \mathfrak{S}$.

A subgroup A of G is σ_3 -subnormal in G if and only if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i} \in \mathfrak{S}$ for all $i = 1, \dots, n$.

(ii) If the second limiting case, when $\sigma = \sigma^1 := \{\{2\}, \{3\}, \{5\}, \dots\}$, $\sigma_0 = \{2\}$ and so $\mathfrak{S}$ is the class of all 2-groups.

A group G is σ^1 -primary (respectively, σ^1 -nilpotent, σ^1 -soluble) if and only if G is a p -group for some prime p (respectively, G is nilpotent, soluble). A subgroup A of G is σ^1 -subnormal in G if and only if A is subnormal in G .

Finally, in view of Remark 0.4 (i), a group G is σ_3 -primary (respectively, σ_3 -nilpotent, σ_3 -soluble) if and only if G is a p -group for some prime p (respectively, G is nilpotent, soluble). A subgroup A of G is σ_3 -subnormal in G if and only if A is subnormal in G .

(iii) Let $|\sigma| = 2$ and $\sigma = \sigma^{\pi, \pi'} := \{\sigma_0, \sigma_1\}$, then G is $\sigma_3^{\pi, \pi'}$ -primary (respectively, $\sigma_3^{\pi, \pi'}$ -soluble, $\sigma_3^{\pi, \pi'}$ -nilpotent) if and only if G is a $\mathfrak{S}$ -group or a σ_1 -group (respectively, every chief factor of G is an $\mathfrak{S}$ -group or a σ_1 -group, $G = H_0 \times H_1$, where

$H_0 \in \mathfrak{Z}$ and H_1 is a Hall σ_1 -subgroup of G by Remark 0.3 (iii)).

A subgroup A if G $\sigma_3^{\pi, \pi'}$ -subnormal in G if and only if there is a chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is an $\mathfrak{Z}$ -group or a σ_1 -group for all $i = 1, \dots, n$.

In particular, G is $\sigma_3^{\pi, \pi'}$ -primary (respectively, $\sigma_3^{\pi, \pi'}$ -soluble, $\sigma_3^{\pi, \pi'}$ -nilpotent) if and only if G is a σ_0 -group or a σ_1 -group (respectively, G is σ_0 -se-parable, G is σ_0 -decomposable, that is, $G = O_\pi(G) \times O_{\pi'}(G)$).

A subgroup A if G $\sigma_3^{\pi, \pi'}$ -subnormal in G if and only if there is a chain $A = A_0 \leq A_1 \leq \dots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is a σ_0 -group or a σ_1 -group for all $i = 1, \dots, n$.

(iv) Now, let $\sigma = \sigma_3^{1, \pi'} := \{\sigma_0, \{p_1\}, \{p_2\}, \dots\}$, where $\sigma_0 = \pi'$ and $\pi = \{p_1, p_2, \dots\}$.

Then G is $\sigma_3^{1, \pi'}$ -primary (respectively, $\sigma_3^{1, \pi'}$ -soluble, $\sigma_3^{1, \pi'}$ -nilpotent) if and only if G is an $\mathfrak{Z}$ -group or a p -group for some $p \in \pi$ (respectively, every chief factor of G is an $\mathfrak{Z}$ -group or p -group for some $p \in \pi$, $G = F(G) \times H$, where $F(G)$ is a nilpotent Hall π -subgroup of G and $H \in \mathfrak{Z}$ by Remark 0.3 (iii)).

(v) Finally, let $\sigma_0 = \{2, 3, 5\}$ and $\mathfrak{Z}$ be the class of all σ_0 -groups G such that every non-abelian composition factor of G is isomorphic to the alternating group A_5 . Then the class $\mathfrak{Z}$ is closed under extensions, epimorphic images and subgroups and it contains all soluble σ_0 -groups. Now let $\sigma = \{\{2, 3, 5\}, \{2, 3, 5\}'\}$. Then the group A_5 is σ_3 -primary and the group A_6 is σ -primary and so σ -nilpotent. On the other hand, $A_6 \notin \mathfrak{Z}$, so A_6 is not σ_3 -primary and hence A_6 is not σ_3 -soluble.

Remark 0.7. The usefulness and universality of the above defined concepts are primarily due to the fact that, in view of Example 0.6 (i), any family of groups $\mathfrak{X}$ is σ_3^* -nilpotent (respectively, σ_3^* -soluble) and any set of subgroups of G is σ_3^{**} -subnormal in G , for some σ_3^* -pair and σ_3^{**} -pair.

In view of Remark 0.7, the following questions are quite natural.

Problem 0.8. Let $\mathfrak{X}$ be an arbitrary collection of groups.

(i) Is there, in some sense, a “minimal” σ_3 -pair such that $\mathfrak{X}$ is σ_3 -nilpotent (respectively, σ_3 -soluble)? And if it is so,

(ii) what then is this minimal σ_3 -pair?

Problem 0.9. Let $\mathfrak{X}$ be an arbitrary collection of groups.

(i) Is there, in some sense, a “minimal” partition σ of $\mathbb{P}$ such that $\mathfrak{X}$ is σ -nilpotent (respectively, σ -soluble)? And if it is so,

(ii) what then is this minimal partition σ ?

Problem 0.10. Let $\mathfrak{X}$ be a set of subgroups of G .

(i) Is there, in some sense, a “minimal” σ_3 -pair such that $\mathfrak{X}$ is σ_3 -subnormal in G ? And if it is so,

(ii) what then is this minimal σ_3 -pair?

Problem 0.11. Let $\mathfrak{X}$ be a set of subgroups of G .

(i) Is there, in some sense, a “minimal” partition σ of $\mathbb{P}$ such that $\mathfrak{X}$ is σ -subnormal in G ? And if it is so,

(ii) what then is this minimal partition σ ?

In this paper, we give a partial answer to these questions.

Finally, note that the results of this work go back to some observations in [11], and partially, they are published in the preprint [12] and in the paper [13].

In [14], interesting computational aspects of the theory of σ -properties were studied. In particular, it was proved that in a permutation group of degree n , in polynomial time $t = t(n)$, one can find the smallest partition σ of $\mathbb{P}$ for which the group G is σ -nilpotent (σ -soluble).

1 The lattices of σ_3 -pairs

We use $\sigma_3(\mathbb{P})$ to denote the set of all σ_3 -pairs. In particular, $\sigma(\mathbb{P})$ denotes the set of all partitions of $\mathbb{P}$.

Definition 1.1. For any two σ_3 -pairs σ_3^* and σ_3^{**} , where $\sigma^* = \{\sigma_i^* \mid i \in I^*\}$ and $\sigma^{**} = \{\sigma_j^{**} \mid j \in I^{**}\}$, we write $\sigma_3^* \leq \sigma_3^{**}$ provided $\mathfrak{Z}^* \subseteq \mathfrak{Z}^{**}$ and for any $i \in I^* \setminus \{0\}$ there exists $j \in I^{**}$ such that $\sigma_i^* \subseteq \sigma_j^{**}$.

In particular, we write $\sigma^* \leq \sigma^{**}$ provided for any $i \in I^*$ there exists $j \in I^{**}$ such that $\sigma_i^* \subseteq \sigma_j^{**}$.

Remark 1.2. In fact, $\sigma_3^* \leq \sigma_3^{**}$ if and only if $\mathfrak{Z}^* \subseteq \mathfrak{Z}^{**}$ and $\sigma^* \leq \sigma^{**}$.

Indeed, since

$$\sigma_0^* = \pi(\mathfrak{Z}^*) \subseteq \pi(\mathfrak{Z}^{**}) = \sigma_0^{**},$$

so from $\sigma_3^* \leq \sigma_3^{**}$ it follows that $\sigma^* \leq \sigma^{**}$.

Let $\mathfrak{Z}^* = \mathfrak{G}$ be the class of all groups and $\mathfrak{Z}^{**} = \mathfrak{G}_2$ the class of all 2-groups.

Proposition 1.3. $\langle \sigma_3(\mathbb{P}), \leq \rangle$ is a partially ordered set with greatest element $\{\mathbb{P}\}_{\mathfrak{Z}^*}$ and least element $\{\{2\}, \{3\}, \{5\}, \dots\}_{\mathfrak{Z}^{**}}$. In particular, $\langle \sigma(\mathbb{P}), \leq \rangle$

is a partially ordered set with greatest element $\{\mathbb{P}\}$ and least element $\{\{2\}, \{3\}, \{5\}, \dots\}$.

Proof. It is clear that $\leq$ is reflexive and, in view of Remark 1.2, transitive.

Now, note that if

$$\sigma_{\mathfrak{Z}^*}^* \leq \sigma_{\mathfrak{Z}^{**}}^{**} \leq \sigma_{\mathfrak{Z}^*}^*,$$

then $\mathfrak{Z}^* \subseteq \mathfrak{Z}^{**} \subseteq \mathfrak{Z}^*$, so $\mathfrak{Z}^* = \mathfrak{Z}^{**}$.

On the other hand, if $i, i_1 \in I^*$ and $j \in I^{**}$ such that

$$\sigma_i^* \subseteq \sigma_j^{**} \subseteq \sigma_{i_1}^*,$$

then $\sigma_i^* = \sigma_{i_1}^* = \sigma_j^{**}$, so for each $i \in I^*$ there is $j \in I^{**}$ such that $\sigma_i = \sigma_j^*$. Similarly, for each $j \in I^{**}$ there is $i \in I^*$ such that $\sigma_j^{**} = \sigma_i^*$. Hence $\sigma^* = \sigma^{**}$, so $\leq$ is antisymmetric.

Thus, $\langle \sigma_{\mathfrak{Z}}(\mathbb{P}), \leq \rangle$ is a partially ordered set with greatest element $\{\mathbb{P}\}_{\mathfrak{Z}^*}$ and least element $\{\{2\}, \{3\}, \{5\}, \dots\}_{\mathfrak{Z}^{**}}$ (see Example 1.6 (i) (ii)).

The proposition is proved.

Proposition 1.4. Let $\{\sigma_{\mathfrak{Z}_j}^i \mid j \in J\}$ be any set of σ_3 -pairs, where $\sigma^j = \{\sigma_{k_j}^i \mid k_j \in K_j\}$ for all $j \in J$. Let $\mathfrak{Z} := \bigcap_{j \in J} \mathfrak{Z}_j$ and let $\{\bigcap_{j \in J} \sigma_{k_j}^i\}$ be the set of all intersections of the form $\bigcap_{j \in J} \sigma_{k_j}^i$. Then the following statements hold:

(i) $\sigma := \bigcap_{j \in J} \sigma^j$ and $\mathfrak{Z}$ form a σ_3 -pair (denoted by $\bigcap_{j \in J} \sigma_{\mathfrak{Z}_j}^i$).

In particular, for any two σ_3 -pairs $\sigma_{\mathfrak{Z}^*}^*$ and $\sigma_{\mathfrak{Z}^{**}}^{**}$, where $\sigma^* = \{\sigma_i^* \mid i \in I^*\}$ and $\sigma^{**} = \{\sigma_j^{**} \mid j \in I^{**}\}$,

$$\sigma_{\mathfrak{Z}^*}^* \cap \sigma_{\mathfrak{Z}^{**}}^{**}$$

is a σ_3 -pair, where

$$\sigma = \{\sigma_i^* \cap \sigma_j^{**} \mid i \in I^*, j \in I^{**}\}$$

and $\mathfrak{Z} = \mathfrak{Z}^* \cap \mathfrak{Z}^{**}$;

(ii) $\sigma_{\mathfrak{Z}} := \bigcap_{j \in J} \sigma_{\mathfrak{Z}_j}^i$ is the greatest of the lower bounds for $\{\sigma_{\mathfrak{Z}_j}^i \mid j \in J\}$ in $\sigma_3(\mathbb{P})$. In particular, $\sigma_3(\mathbb{P})$ is a complete lattice;

(iii) if $\mathfrak{X}$ is $\sigma_{\mathfrak{Z}_j}^i$ -nilpotent (respectively, $\sigma_{\mathfrak{Z}_j}^i$ -soluble) for all $j \in J$, then $\mathfrak{X}$ is σ_3 -nilpotent (respectively, σ_3 -soluble);

(iv) if $\mathfrak{X} \subseteq L(G)$ and $\mathfrak{X}$ is $\sigma_{\mathfrak{Z}_j}^i$ -subnormal in G for all $j \in J$, then $\mathfrak{X}$ is σ_3 -subnormal in G .

Proof. See the proof of Proposition 1.6.7 in [12].

Corollary 1.5. Let $\{\sigma^j \mid j \in J\}$ be any set partitions of $\mathbb{P}$, where $\sigma^j = \{\sigma_{k_j}^i \mid k_j \in K_j\}$ for all $j \in J$. Let $\{\bigcap_{j \in J} \sigma_{k_j}^i\}$ be the set of all intersections of the form $\bigcap_{j \in J} \sigma_{k_j}^i$. Then the following statements hold:

(i) $\sigma := \bigcap_{j \in J} \sigma^j$ is a partition of $\mathbb{P}$ (denoted by $\bigcap_{j \in J} \sigma^j$).

In particular, for any two partition σ_3 -pairs σ^* and σ^{**} of $\mathbb{P}$, where $\sigma^* = \{\sigma_i^* \mid i \in I^*\}$ and $\sigma^{**} = \{\sigma_j^{**} \mid j \in I^{**}\}$,

$$\sigma^* \cap \sigma^{**}$$

is a partition of $\mathbb{P}$, where

$$\sigma = \{\sigma_i^* \cap \sigma_j^{**} \mid i \in I^*, j \in I^{**}\};$$

(ii) $\sigma := \bigcap_{j \in J} \sigma^j$ is the greatest of the lower bounds for $\{\sigma^j \mid j \in J\}$ in $\sigma(\mathbb{P})$. In particular, $\sigma(\mathbb{P})$ is a complete lattice;

(iii) if $\mathfrak{X}$ is σ^j -nilpotent (respectively, σ^j -soluble for all $j \in J$), then $\mathfrak{X}$ is σ -nilpotent (respectively, σ -soluble);

(iv) if $\mathfrak{X} \subseteq L(G)$ and $\mathfrak{X}$ is σ^j -subnormal in G for all $j \in J$, then $\mathfrak{X}$ is σ -subnormal in G .

2 The $\mathfrak{X}$ -nilpotent and the $\mathfrak{X}$ -soluble σ_3 -pairs

We write $\mathfrak{N}_{\sigma_3}$ (respectively, $\mathfrak{S}_{\sigma_3}$) to denote the class of all σ_3 -nilpotent (respectively, σ_3 -soluble) groups; $\mathfrak{N}$ (respectively, $\mathfrak{S}$) is the class of all nilpotent (respectively, soluble) groups.

Definition 2.1. (i) If $\Sigma = \{\sigma_{\mathfrak{Z}_j}^i \mid j \in J\}$ is the set of all σ_3 -pairs such that $\mathfrak{X} \subseteq \mathfrak{N}_{\sigma_{\mathfrak{Z}_j}^i}$ (respectively, $\mathfrak{X} \subseteq \mathfrak{S}_{\sigma_{\mathfrak{Z}_j}^i}$) for all $j \in J$, then we write $\Sigma_{n, \Sigma}(\mathfrak{X})$ (respectively, $\Sigma_{s, \Sigma}(\mathfrak{X})$) to denote the σ_3 -pair $\bigcap_{j \in J} \sigma_{\mathfrak{Z}_j}^i$.

(ii) If $\Sigma = \{\sigma^j \mid j \in J\}$ is the set of all partitions of $\mathbb{P}$ such that $\mathfrak{X} \subseteq \mathfrak{N}_{\sigma^j}$ (respectively, $\mathfrak{X} \subseteq \mathfrak{S}_{\sigma^j}$) for all $j \in J$, then we write $\Sigma_n(\mathfrak{X})$ (respectively, $\Sigma_s(\mathfrak{X})$) to denote the partition $\bigcap_{j \in J} \sigma^j$ of $\mathbb{P}$.

By Proposition 1.4 (iii), $\Sigma_{n, \Sigma}(\mathfrak{X}) \in \Sigma$ (respectively, $\Sigma_{s, \Sigma}(\mathfrak{X}) \in \Sigma$) and $\Sigma_{n, \Sigma}(\mathfrak{X})$ (respectively, $\Sigma_{s, \Sigma}(\mathfrak{X})$) is the smallest element in Σ .

By Corollary 1.5 (iii), $\Sigma_n(\mathfrak{X}) \in \Sigma$ (respectively, $\Sigma_s(\mathfrak{X}) \in \Sigma$) and $\Sigma_n(\mathfrak{X})$ (respectively, $\Sigma_s(\mathfrak{X})$) is the smallest element in Σ .

Definition 2.2. (i) We say that $\Sigma_{n,\mathfrak{Z}}(\mathfrak{X})$ is the $\mathfrak{X}$ -nilpotent $\sigma_{\mathfrak{Z}}$ -pair and $\Sigma_{s,\mathfrak{Z}}(\mathfrak{X})$ is the $\mathfrak{X}$ -soluble $\sigma_{\mathfrak{Z}}$ -pair.

(ii) We say that $\Sigma_n(\mathfrak{X})$ is the $\mathfrak{X}$ -nilpotent partition and $\Sigma_s(\mathfrak{X})$ is the $\mathfrak{X}$ -soluble partition of $\mathbb{P}$.

It is clear that $\Sigma_{n,\mathfrak{Z}}(\emptyset) = \sigma^1 = \Sigma_{s,\mathfrak{Z}}(\emptyset)$ and $\Sigma_n(\emptyset) = \sigma^1 = \Sigma_s(\emptyset)$.

If $\mathfrak{X} = \{G\}$, we write $\Sigma_{n,\mathfrak{Z}}(G)$, $\Sigma_{s,\mathfrak{Z}}(G)$ and $\Sigma_n(G)$, $\Sigma_s(G)$.

If $G \notin \mathfrak{N}_{\sigma_{\mathfrak{Z}}}$ but every proper subgroup of G belongs to $\mathfrak{N}_{\sigma_{\mathfrak{Z}}}$, then G is called an $\mathfrak{N}_{\sigma_{\mathfrak{Z}}}$ -critical or a minimal non- $\sigma_{\mathfrak{Z}}$ -nilpotent group. If G is not nilpotent but every proper subgroup of G is nilpotent, then G is said to be a Schmidt group.

Lemma 2.3. Suppose that G is a minimal non- $\sigma_{\mathfrak{Z}}$ -nilpotent group, then G is either a Schmidt group or a non-abelian simple minimal non- $\mathfrak{Z}$ -group.

Proof. Let $\sigma = \{\sigma_i \mid i \in I\}$. It is clear that G is not a $\sigma_{\mathfrak{Z}}$ -primary group, so G is not a σ_i -group for all $i \neq 0$.

First assume that G is a σ_0 -group. Then G is a minimal non- $\mathfrak{Z}$ -group, since every proper subgroup of G is $\sigma_{\mathfrak{Z}}$ -nilpotent and so it is $\sigma_{\mathfrak{Z}}$ -primary. Moreover, G is a non-abelian simple group since $\mathfrak{Z}$ is a complete class of groups and so it is closed under extensions.

Now assume that G is not a σ_0 -group, so G is not a σ_i -group for all $i \in I$. In view of Remark 0.4(iii), a group is σ -nilpotent if and only if G is σ_i -decomposable for all $i \in I$. Therefore, for some i , G is a minimal non- σ_i -decomposable group and so G is a Schmidt group by [15]. $\square$

Corollary 2.4. If G is a minimal non- σ -nilpotent group, then G is a Schmidt group.

Now, returning to Problem 0.8, we see that:

(i) a “minimal” $\sigma_{\mathfrak{Z}}$ -pair for which the family $\mathfrak{X}$ is $\sigma_{\mathfrak{Z}}$ -nilpotent (respectively, $\sigma_{\mathfrak{Z}}$ -soluble) exists and $\sigma_{\mathfrak{Z}}$ coincides with $\Sigma_{n,\mathfrak{Z}}(\mathfrak{X})$ (respectively, with $\Sigma_{s,\mathfrak{Z}}(\mathfrak{X})$);

(ii) a “minimal” partition σ of $\mathbb{P}$ for which the family $\mathfrak{X}$ is σ -nilpotent (respectively, σ -soluble) exists and σ coincides with $\Sigma_n(\mathfrak{X})$ (respectively, with $\Sigma_s(\mathfrak{X})$).

Our next goal is to give an answer to the second part of Problem 0.8.

Definition 2.5. Let $\mathfrak{X}$ be a (possibly empty) family of groups. Then we define the sets of groups $S = S(\mathfrak{X})$, $S^* = S^*(\mathfrak{X})$ and $S^0 = S^0(\mathfrak{X})$ as follows:

(i) $A \in S$ if and only if A is a Schmidt group and there is a group $G \in \mathfrak{X}$ and a section H/K of G such that $A \simeq H/K$;

(ii) $A \in S^*$ if and only if A is a non-abelian simple group and there is a group $G \in \mathfrak{X}$ and a section H/K of G such that $A \simeq H/K$.

(iii) $A \in S^0$ if and only if $A \in S$ and $\pi(A) \cap \pi(S^*) \neq \emptyset$.

If $\mathfrak{X} = \{G\}$, we write $S(G)$ (respectively, $S^*(G)$, $S^0(G)$).

It is clear that if $\mathfrak{X}$ is nilpotent (respectively, soluble), then $S(\emptyset) = \emptyset$ (respectively, $S^*(\emptyset) = \emptyset$). In particular, $S(\emptyset) = \emptyset = S^*(\emptyset)$.

Definition 2.6. (i) For any set of Schmidt groups $S = \{A_k \mid k \in K\}$ and any set of non-abelian simple groups S^* we define:

(i) a partition $\sigma_{S,S^*} = \{\sigma_i \mid i \in I = \{0\} \cup I\}$ of $\mathbb{P}$, where $\sigma_0 = \pi(S^* \cup S^0)$ and for any σ_i such that $\sigma_i \cap \sigma_0 = \emptyset$ and $p \neq q \in \sigma_i$ there exist groups $A_{k_1}, \dots, A_{k_t}$ in S such that $p \in \pi(A_{k_1})$, $q \in \pi(A_{k_t})$ and $\pi(A_{k_{j-1}}) \cap \pi(A_{k_j}) \neq \emptyset$ for $j = 2, \dots, t$;

(ii) a partition $\sigma_{S^*} = \{\sigma_j \mid j \in J = \{0\} \cup J\}$ of $\mathbb{P}$, where $\sigma_0 = \pi(S^*)$ and $\sigma_j = \{p\}$, where p is a prime, for all $j \neq 0$;

(iii) a partition $\sigma_S = \{\sigma_k \mid k \in K\}$ of $\mathbb{P}$, where for any σ_i such that $|\sigma_i| > 1$ and $p, q \in \sigma_i$ there exist groups $A_{k_1}, \dots, A_{k_t}$ in S such that $p \in \pi(A_{k_1})$, $q \in \pi(A_{k_t})$ and $\pi(A_{k_{j-1}}) \cap \pi(A_{k_j}) \neq \emptyset$ for $j = 2, \dots, t$;

(iv) a complete set of groups $\mathfrak{Z}_{S^*}$ such that it is the intersection of all complete sets of groups containing $S^* \cup S^0$.

Lemma 2.7. Let $\mathfrak{X}$ be a non-soluble family of groups. Let $S = S(\mathfrak{X})$, $S^* = S^*(\mathfrak{X})$ and $S^0 = S^0(\mathfrak{X})$. Then σ_{S,S^*} , $\mathfrak{Z}_{S^0}$ and σ_{S^*} , $\mathfrak{Z}_{S^*}$ form $\sigma_{\mathfrak{Z}}$ -pairs.

Proof. First note that $\sigma_0 = \pi(S^* \cup S^0) = \pi(\mathfrak{Z}_{S^*})$ since the class of all $\pi(S^* \cup S^0)$ -groups is a complete set of subgroups and so, in fact, we have only to show that every soluble σ_0 -group G belongs to $\mathfrak{Z}_{S^*}$.

Let $\mathfrak{Z}^*$ be any complete set of groups containing $S^* \cup S^0$. Then every chief factor of G belongs to $\mathfrak{Z}^*$ since $\sigma_0 \subseteq \pi(\mathfrak{Z}^*)$ and $\mathfrak{Z}^*$ is hereditary, so G belongs to $\mathfrak{Z}^*$ since $\mathfrak{Z}^*$ is closed

with respect to the extension of its groups. Hence $G \in \mathfrak{S}_{S^*}$.

It can be proved similarly, that σ_{S^*} and $\mathfrak{S}_{S^*}$ form a $\sigma_{\mathfrak{S}}$ -pair. $\square$

Example 2.8. (i) By definition, in the case when $S = \emptyset$, we have $\sigma_S = \{\{2\}, \{3\}, \{5\}, \dots\}$.

(ii) If $A = A_p \rtimes A_q$ is a Schmidt group and $S = \{A\}$, then $\sigma_S = \{\{p, q\}, \{r\}, \{s\}, \dots\}$, where $\{r, s, \dots\} \cap \{p, q\} = \emptyset$.

(iii) Let S be the set of all Schmidt groups. Then $\sigma_S = \{\mathbb{P}\}$. Indeed, let $\sigma_S = \{\sigma_k \mid k \in K\}$ and let $p \in \sigma_j$. Then for every prime $q \neq p$ there exists a Schmidt group A such that $\pi(A) = \{p, q\}$, so $q \in \sigma_j$. Hence, in fact, $\mathbb{P} \subseteq \sigma_j$.

Lemma 2.9 [16, Proposition 2.7]. *The class $\mathfrak{S}_{\sigma_{\mathfrak{S}}}$ is complete.*

Theorem 2.10. *Let $\mathfrak{X}$ be a non-soluble family of groups and let $S^* = S^*(\mathfrak{X})$. Let $\sigma^* = \sigma_{S^*}$ and $\mathfrak{S}^* = \mathfrak{S}_{S^*}$. Then the following statements hold.*

- (i) *If $\mathfrak{X}$ is $\sigma_{\mathfrak{S}}$ -soluble, then $\sigma_{S^*}^* \leq \sigma_{\mathfrak{S}}$.*
- (ii) *$\mathfrak{X}$ is $\sigma_{S^*}^*$ -soluble.*
- (iii) $\Sigma_{s, \mathfrak{S}}(\mathfrak{X}) = \sigma_{S^*}^* = \Sigma_{s, \mathfrak{S}}(S^*)$.

Proof. (i) Let $\sigma = \{\sigma_i \mid i \in I\}$ and

$$\sigma_{S^*} = \{\sigma_j^* \mid j \in J = \{0\} \cup J\}.$$

Assume that $\mathfrak{X}$ is $\sigma_{\mathfrak{S}}$ -soluble. We show that $\sigma_{S^*}^* \leq \sigma_{\mathfrak{S}}$.

First we show that $\mathfrak{S}_{S^*} \subseteq \mathfrak{S}$. In view of Lemma 2.9, every group $G \in S^*$ is $\sigma_{\mathfrak{S}}$ -soluble and so it is $\sigma_{\mathfrak{S}}$ -primary. Hence G is a σ_i -group for some i . Assume that $i \neq 0$, then $2 \notin \sigma_i$, so G is soluble by the Feit-Thompson theorem on solubility groups of odd order, a contradiction. Therefore $i = 0$ and so $G \in \mathfrak{S}$. Hence $S^* \subseteq \mathfrak{S}$ which implies that $\mathfrak{S}_{S^*} \subseteq \mathfrak{S}$.

Finally, for every $j \in J \setminus \{0\}$ we have $|\sigma_j^*| = 1$, by definition of the partition $\sigma_{S^*}^*$, so for some r it follows that so $\sigma_j^* \subseteq \sigma_r$. Hence $\sigma_{S^*}^* \leq \sigma_{\mathfrak{S}}$.

(ii) Now we show that $\mathfrak{X}$ is $\sigma_{S^*}^*$ -soluble.

Assume that some section H/K of some group $G \in \mathfrak{X}$ is not $\sigma_{S^*}^*$ -soluble and let A be a minimal non- $\sigma_{S^*}^*$ -soluble subgroup of H/K . Then A is a non-abelian simple group, so $A \in S^* \subseteq \mathfrak{S}^*$ and hence A is $\sigma_{S^*}^*$ -soluble, a contradiction. Hence (ii) holds.

(iii) From Proposition 1.7 and Part (i) it follows that

$$\sigma_{S^*}^* \leq \Sigma_{s, \mathfrak{S}}(\mathfrak{X}).$$

On the other hand, from Part (ii) it follows that $\Sigma_{s, \mathfrak{S}}(\mathfrak{X}) \leq \sigma_{S^*}^*$. Hence $\Sigma_{s, \mathfrak{S}}(\mathfrak{X}) = \sigma_{S^*}^*$.

Finally, note that $S^* = S^*(\mathfrak{X}) = S^*(S^*)$, so we have (iii). $\square$

Theorem 2.11 [13, Theorem 3.12]. *Let $\mathfrak{X}$ be a non-soluble family of groups and let $S = S(\mathfrak{X})$, $S^* = S^*(\mathfrak{X})$. Let $\sigma^* = \sigma_{S^*}$ and $\mathfrak{S}^* = \mathfrak{S}_{S^*}$. Then the following statements hold.*

- (i) *If $\mathfrak{X}$ is $\sigma_{\mathfrak{S}}$ -nilpotent, then $\sigma_{S^*}^* \leq \sigma_{\mathfrak{S}}$.*
- (ii) *$\mathfrak{X}$ is $\sigma_{S^*}^*$ -nilpotent.*
- (iii) $\Sigma_{n, \mathfrak{S}}(\mathfrak{X}) = \sigma_{S^*}^* = \Sigma_{n, \mathfrak{S}}(S \cup S^*)$.

Corollary 2.12. *Let $\mathfrak{X}$ be a soluble non-nilpotent family of groups and let $S = S(\mathfrak{X})$. Let $\sigma^* = \sigma_S$. Then the following statements hold:*

- (i) *if $\mathfrak{X}$ is σ -nilpotent, then $\sigma^* \leq \sigma$.*
- (ii) *$\mathfrak{X}$ is σ^* -nilpotent.*
- (iii) $\Sigma_n(\mathfrak{X}) = \sigma^* = \Sigma_n(S)$.

3 The (G, A) -subnormal $\sigma_{\mathfrak{S}}$ -pair

Definition 3.1. (i) Let A be a subgroup of G and let $\{\sigma_{\mathfrak{S}_j}^j \mid j \in J\}$ be the set of all of $\sigma_{\mathfrak{S}}$ -pairs, where $\sigma^j = \{\sigma_{k_j}^j \mid k_j \in K_j\}$ for all $j \in J$, such that A is $\sigma_{\mathfrak{S}_j}^j$ -subnormal in G for all $j \in J$. Then we write $\Sigma_{sn, \mathfrak{S}}(G, A)$ to denote the $\sigma_{\mathfrak{S}}$ -pair $\bigcap_{j \in J} \sigma_{\mathfrak{S}_j}^j$.

(ii) Let A be a subgroup of G and let $\{\sigma^j \mid j \in J\}$ be the of partitions of $\mathbb{P}$, where $\sigma^j = \{\sigma_{k_j}^j \mid k_j \in K_j\}$ for all $j \in J$, such that A is σ^j -subnormal in G for all $j \in J$. Then we write $\Sigma_{sn}(G, A)$ to denote the partition $\bigcap_{j \in J} \sigma^j$ of $\mathbb{P}$.

By Proposition 1.7 (iv), $\Sigma_{sn, \mathfrak{S}}(G, A)$ is the smallest element in $\{\sigma_{\mathfrak{S}_j}^j \mid j \in J\}$, that is, $\Sigma_{sn, \mathfrak{S}}(G, A) \in \{\sigma_{\mathfrak{S}_j}^j \mid j \in J\}$ and $\Sigma_{sn, \mathfrak{S}}(G, A) \leq \sigma_{\mathfrak{S}_j}^j$ for all $j \in J$.

We say that $\Sigma_{sn, \mathfrak{S}}(G, A)$ is the (G, A) -subnormal $\sigma_{\mathfrak{S}}$ -pair.

By Corollary 1.8 (iv), $\Sigma_{sn}(G, A)$ is the smallest element in $\{\sigma^j \mid j \in J\}$, that is,

$$\Sigma_{sn}(G, A) \in \{\sigma^j \mid j \in J\}$$

and $\Sigma_{sn}(G, A) \leq \sigma^j$ for all $j \in J$.

We say that $\Sigma_{sn}(G, A)$ is the (G, A) -subnormal partition of $\mathbb{P}$.

Now, returning to Problem 0.9 (i) we see that a “minimal” σ_3 -pair for which A is σ_3 -subnormal in G exists and σ_3 coincides with $\Sigma_{sn,3}(G, A)$.

Thus, Proposition 1.7 (iv) gives an answer to the first part of Problem 0.9.

Now we shall give a partial answer to the second part of Problem 0.9.

We say that $A = A_0 < A_1 < \dots < A_{t-1} < A_t = G$ is a maximal σ_3 -subnormal chain of G provided A_i is a maximal σ_3 -subnormal subgroup of A_{i+1} for all $i = 0, 1, \dots, t-1$.

Lemma 3.2. *A proper subgroup A of G is a maximal σ_3 -subnormal subgroup of G if and only if either $G/A_G = G/A$ is a non- σ_3 -primary simple group or A is a maximal subgroup of G and G/A_G is a σ_3 -primary group.*

Proof. First suppose that A is a maximal σ_3 -subnormal subgroup of G . Assume that A is normal in G and $G/A = G/A_G$ is a simple group. If G/A is a group of prime order, then A is a maximal subgroup of G and $G/A_G = G/A$ is a σ_3 -primary group. Now assume that $G/A = G/A_G$ is a non-abelian simple group and let $A \leq M$, where M is a maximal subgroup of G . If $G/A_G = G/A$ is a σ_3 -primary group, then G/M_G is a σ_3 -primary group and so M is σ_3 -subnormal in G . The maximality of A implies that $A = M$.

Finally, if either $G/A_G = G/A$ is a non- σ -primary simple group or A is a maximal non-normal subgroup of G and G/A_G is a σ -primary group, then A is a maximal σ -subnormal subgroup of G . $\square$

Theorem 3.3. *Let A be a σ_3 -subnormal subgroup of G , where $\sigma_3 = \Sigma_{sn,3}(G, A)$. Let*

$$A = A_0 < A_1 < \dots < A_{t-1} < A_t = G$$

be a maximal σ_3 -subnormal chain of G with as minimal as possible set of i such that $A_{i-1} \not\trianglelefteq A_i$, and let

$$X = \{A_i / (A_{i-1})_{A_i} \mid A_{i-1} \neq (A_{i-1})_{A_i}\}.$$

Then $\sigma_3 = \Sigma_{n,3}(X)$.

Proof. Let $\sigma_3^* = \Sigma_{n,3}(\mathfrak{X})$.

If $\mathfrak{X} = \emptyset$, then A is subnormal in G and so $\sigma = \sigma^1$. On the other hand, $\sigma^* = \Sigma_{n,3}(\emptyset) = \sigma^1$.

Now assume that $\mathfrak{X} \neq \emptyset$. First we show that the subgroup A is σ_3^* -subnormal in G . Indeed, in a maximal σ_3 -subnormal chain

$$A = A_0 < A_1 < \dots < A_{t-1} < A_t = G$$

of G for every i such that $A_k \neq (A_{k-1})_{A_k}$ the factor $A_k / (A_{k-1})_{A_k}$ belongs to $\mathfrak{X}$, so this factor is σ_3^* -nilpotent and hence A is σ_3^* -subnormal. It follows that $\sigma_3 \leq \sigma_3^*$.

In view of Lemma 3.2, for a maximal σ_3 -subnormal chain $A = A_0 < A_1 < \dots < A_{t-1} < A_t = G$ of G every factor $A_{i-1} / (A_{i-1})_{A_i}$, where i is such that $A_{i-1} \neq (A_{i-1})_{A_i}$, is σ_3 -primary and so σ_3 -nilpotent. Therefore $\mathfrak{X}$ is σ_3 -nilpotent, so $\sigma_3^* \leq \sigma_3$ by the definition of

$$\sigma_3^* = \Sigma_{n,3}(\mathfrak{X}).$$

Therefore $\sigma_3 = \sigma_3^*$. $\square$

The following theorem is proved similarly.

Theorem 3.4. *Let A be a σ -subnormal subgroup of G , where $\sigma = \Sigma_{sn}(G, A)$. Let*

$$A = A_0 < A_1 < \dots < A_{t-1} < A_t = G$$

be a maximal σ -subnormal chain of G with as minimal as possible set of i such that $A_{i-1} \not\trianglelefteq A_i$, and let

$$X = \{A_i / (A_{i-1})_{A_i} \mid A_{i-1} \neq (A_{i-1})_{A_i}\}.$$

Then $\sigma = \Sigma_n(X)$.

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- Информация об авторах**
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