

О $\sigma_{\mathfrak{S}}$ -СУБНОРМАЛЬНЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУПП

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ON $\sigma_{\mathfrak{S}}$ -SUBNORMAL SUBGROUPS OF FINITE GROUPS

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Аннотация. В данной работе: G – конечная группа; $\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$ – некоторое разбиение множества всех простых чисел \mathbb{P} , где $0 \in I$; \mathfrak{S} – класс конечных σ_0 -групп, который замкнут относительно расширений, эпиморфных образов и подгрупп и который содержит все конечные разрешимые σ_0 -группы. Группа G называется: (i) $\sigma_{\mathfrak{S}}$ -примарной, если G является конечной σ_i -группой для некоторого $i \in I$ и $G \in \mathfrak{S}$, если $i = 0$; (ii) $\sigma_{\mathfrak{S}}$ -нильпотентной, если G является прямым произведением $\sigma_{\mathfrak{S}}$ -первичных групп. Подгруппа A группы G называется $\sigma_{\mathfrak{S}}$ -субнормальной в G , если существует цепь подгрупп $A = A_0 \leq A_1 \leq \dots \leq A_t = G$ такая, что либо $A_{i-1} \trianglelefteq A_i$, либо $A_i / (A_{i-1})_{A_i}$ является $\sigma_{\mathfrak{S}}$ -примарной для всех $i = 1, \dots, t$. В данной работе мы изучаем $\sigma_{\mathfrak{S}}$ -нильпотентные группы, $\sigma_{\mathfrak{S}}$ -субнормальные подгруппы и соотношения между $\sigma_{\mathfrak{S}}$ -нильпотентностью и $\sigma_{\mathfrak{S}}$ -субнормальностью в группах. В частности, мы доказываем, что группа G является $\sigma_{\mathfrak{S}}$ -нильпотентной в том и только в том случае, когда все ее подгруппы $\sigma_{\mathfrak{S}}$ -субнормальны.

Ключевые слова: конечная группа, $\sigma_{\mathfrak{S}}$ -примарная группа, $\sigma_{\mathfrak{S}}$ -нильпотентная группа, $\sigma_{\mathfrak{S}}$ -разрешимая группа, $\sigma_{\mathfrak{S}}$ -субнормальная подгруппа.

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Abstract. In this paper: G is a finite group; $\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$ is some partition of the set of all primes \mathbb{P} , where $0 \in I$; \mathfrak{S} is a class of finite σ_0 -groups which is closed under extensions, epimorphic images and subgroups and which contains all finite soluble σ_0 -groups. A group G is said to be: (i) $\sigma_{\mathfrak{S}}$ -primary provided G is a finite σ_i -group for some $i \in I$ and $G \in \mathfrak{S}$ if $i = 0$; (ii) $\sigma_{\mathfrak{S}}$ -nilpotent if G is the direct product of $\sigma_{\mathfrak{S}}$ -primary groups. A subgroup A of G is said to be $\sigma_{\mathfrak{S}}$ -subnormal in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \dots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{S}}$ -primary for all $i = 1, \dots, t$. In this paper, we study $\sigma_{\mathfrak{S}}$ -nilpotent groups, $\sigma_{\mathfrak{S}}$ -subnormal subgroups and relations between $\sigma_{\mathfrak{S}}$ -nilpotency and $\sigma_{\mathfrak{S}}$ -subnormality in the groups. In particular, we prove that a group G is $\sigma_{\mathfrak{S}}$ -nilpotent if and only if all its subgroups are $\sigma_{\mathfrak{S}}$ -subnormal.

Keywords: finite group, $\sigma_{\mathfrak{S}}$ -primary group, $\sigma_{\mathfrak{S}}$ -nilpotent group, $\sigma_{\mathfrak{S}}$ -soluble group, $\sigma_{\mathfrak{S}}$ -subnormal subgroup.

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1 Introduction

Throughout this paper, all groups are finite, and G always denotes a finite group.

In what follows,

$$\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$$

is some partition of the set of all primes \mathbb{P} , where $0 \in I$; \mathfrak{S} is a class of σ_0 -groups which is closed

under extensions, epimorphic images and subgroups and which contains all soluble σ_0 -groups. Moreover, $\pi \subseteq \mathbb{P}$, $\pi' = \mathbb{P} \setminus \pi$ and

$$\Pi \subseteq \sigma, \Pi' = \sigma \setminus \Pi.$$

Definition 1.1. We say that a group G is:

(i) $\sigma_{\mathfrak{S}}$ -primary provided G is a σ_i -group for some $i \in I$ and $G \in \mathfrak{S}$ if $i = 0$;

(ii) $\sigma_{\mathfrak{S}}$ -nilpotent if G is the direct product of $\sigma_{\mathfrak{S}}$ -primary groups;

(iii) $\sigma_{\mathfrak{S}}$ -soluble if every chief factor of G is $\sigma_{\mathfrak{S}}$ -primary.

Definition 1.2. We say that a subgroup A of G is $\sigma_{\mathfrak{S}}$ -subnormal in G if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G \quad (1.1)$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\sigma_{\mathfrak{S}}$ -primary for all $i = 1, \dots, t$.

Remark 1.3. (i) In what follows, we always omit the symbol \mathfrak{S} in all definitions and notations in the case when \mathfrak{S} is the class of all σ_0 -groups.

Thus we say, for example, that G is [1]–[3]: (i) σ -primary if G is a σ_i -group for some $i \in I$; (ii) σ -nilpotent if G is the direct product of σ -primary groups; (iii) σ -soluble if every chief factor of G is σ -primary.

We also say that a subgroup A of G is σ -subnormal in G [1]–[3] if G has a subgroup chain (1.1) such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ -primary for all $i = 1, \dots, t$.

(ii) If $2 \notin \sigma_0$, then every σ_0 -group is soluble by the Feit-Thompson’s theorem, so \mathfrak{S} coincides with the class of all σ_0 -groups since every soluble σ_0 -group belongs to \mathfrak{S} by the definition.

In the literature, one can find several special versions of $\sigma_{\mathfrak{S}}$ -subnormality. The next results describe the first two of them.

Recall that a subgroup A of G is called \mathfrak{F} -subnormal in G in the sense of Kegel [4] or K - \mathfrak{S} -subnormal in G [5, 6.1.4] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ belongs to \mathfrak{F} for all $i = 1, \dots, t$.

Theorem 1.4. If a class of groups \mathfrak{F} is closed under extensions, epimorphic images and subgroups, then the set $\mathcal{L}_{\mathfrak{F}sn}(G)$ ($G \neq 1$), of all K - \mathfrak{F} -subnormal subgroups of G , coincides with the set of all $\sigma_{\mathfrak{S}}$ -subnormal subgroups of G , where $\mathfrak{S} = \mathfrak{F}$ and

$$\sigma = \{\sigma_0, \{p_1\}, \dots, \{p_n\}, \dots\}, \quad \sigma_0 = \bigcup_{X \in \mathfrak{S}} \pi(X),$$

and

$$\{p_1, \dots, p_n, \dots\} = \mathbb{P} \setminus \sigma_0.$$

Let

$$\sigma = \{\sigma_i \mid i \in I \subseteq \{0\} \cup \mathbb{N}\}$$

be some partition of \mathbb{P} , where $0 \in I$. Let $\sigma_0 \in \Pi \subseteq \sigma$ and \mathfrak{S} be a class of σ_0 -groups which is closed under extensions, epimorphic images and

subgroups. Then G is said to be $\Pi_{\mathfrak{S}}$ -primary [6] provided either $G \in \mathfrak{S}$ or G is a σ_i -group for some $\sigma_i \in \Pi \setminus \{\sigma_0\}$. A subgroup A of G is said to be $\Pi_{\mathfrak{S}}$ -subnormal in G [6] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is $\Pi_{\mathfrak{S}}$ -primary for all $i = 1, \dots, t$.

Our next observation shows that the $\Pi_{\mathfrak{S}}$ -subnormality is also a special case of the $\sigma_{\mathfrak{S}}$ -subnormality.

Theorem 1.5. A subgroup A is $\Pi_{\mathfrak{S}}$ -subnormal in G if and only if A is $\sigma_{\mathfrak{S}}$ -subnormal in G , where

$$\sigma = (\Pi \setminus \{\sigma_0\}) \cup \{\{\sigma_0^*\}\} \cup \{\{p_1\}, \dots, \{p_n\}, \dots\},$$

$$\sigma_0^* = \bigcup_{X \in \mathfrak{S}} \pi(X),$$

$$\text{and } \{p_1, \dots, p_n, \dots\} = \mathbb{P} \setminus (\sigma_0^* \cup \bigcup_{\sigma_i \in \Pi \setminus \{\sigma_0\}} \sigma_i).$$

A group G is said to be: (i) π -decomposable if $G = O_{\pi}(G) \times O_{\pi'}(G)$; (ii) π -special [3], [7] if

$$G = F(G) \times O_{\pi}(G) = O_{p_1}(G) \times \dots \times O_{p_t}(G) \times O_{\pi}(G),$$

where $p_1, \dots, p_t \in \pi$.

In order to consider further useful examples, we first introduce the following three special partitions of the set \mathbb{P} :

$$(i) \quad \sigma = \sigma^1 := \{\{2\}, \{3\}, \{5\}, \dots\};$$

$$(ii) \quad \sigma = \sigma^{\pi} := \{\pi, \pi'\};$$

$$(iii) \quad \sigma = \sigma^{1, \pi'} := \{\{p_1\}, \dots, \{p_n\}, \dots, \pi'\}, \quad \text{where } \pi = \{p_1, \dots, p_n, \dots\}.$$

Example 1.6. (i) In the limiting case, when $\sigma = \{\mathbb{P}\}$, $\sigma_0 = \mathbb{P}$, all soluble groups are contained in \mathfrak{S} , and every group $G \in \mathfrak{S}$ is $\sigma_{\mathfrak{S}}$ -primary and so $\sigma_{\mathfrak{S}}$ -nilpotent.

(ii) In the second limiting case, when $\sigma = \sigma^1$, $\sigma_0 = \{p\}$ for some prime p and so \mathfrak{S} is the class of all p -groups. Thus, G is $\sigma_{\mathfrak{S}}$ -primary (respectively, $\sigma_{\mathfrak{S}}$ -nilpotent, $\sigma_{\mathfrak{S}}$ -soluble) if and only if G is a q -group for some prime q (respectively, G is nilpotent, soluble).

A subgroup A of G is $\sigma_{\mathfrak{S}}$ -subnormal in G if and only if A is subnormal in G .

(iii) Let $\sigma = \{\sigma_0, \sigma_1\} = \sigma^{\pi}$, where $\sigma_0 = \pi'$ and $\sigma_1 = \pi$.

Then G is $\sigma_{\mathfrak{S}}$ -primary (respectively, $\sigma_{\mathfrak{S}}$ -nilpotent) if and only if G is either an \mathfrak{S} -group or a σ_1 -group (respectively, $G = H_0 \times H_1$, where $H_0 \in \mathfrak{S}$ and H_1 is a Hall σ_1 -subgroup of G by Lemma 2.1 below). Moreover, G is $\sigma_{\mathfrak{S}}$ -soluble if

and only if every chief factor of G is either an \mathfrak{S} -group or σ_1 -group.

If $2 \in \sigma_1$, then G is σ_3 -primary (respectively, σ_3 -nilpotent, σ_3 -soluble) if and only if G is σ -primary (respectively, σ -nilpotent, σ -soluble). Moreover, in this case a subgroup A of G is σ_3 -subnormal in G if and only if A is σ -subnormal in G .

(iv) Now, let

$$\sigma = \{\sigma_0, \{p_1\}, \dots, \{p_n\}, \dots\} = \sigma^{1, \pi'}$$

where $\sigma_0 = \pi'$ and

$$\pi = \{p_1, \dots, p_n, \dots\}.$$

First assume that $2 \in \pi'$. Then G is σ_3 -primary (respectively, σ_3 -nilpotent, σ_3 -soluble) if and only if G is a \mathfrak{S} -group or a p -group for some prime $p \in \pi$ (respectively, $G = F(G) \times H$, where $F(G)$ is a Hall π -subgroup and $H \in \mathfrak{S}$ by Lemma 2.1 below).

Finally, if $2 \in \pi$, then G is σ_3 -primary (respectively, σ_3 -nilpotent, σ_3 -soluble) if and only if G is σ -primary (respectively, σ -nilpotent, σ -soluble).

Let \mathfrak{F} be a class of groups containing all identity groups. Then \mathfrak{F} is said to be:

(i) a *formation* if the following two conditions hold: (1) \mathfrak{F} is closed under taking subdirect products, that is, $G/(N \cap M) \in \mathfrak{F}$ whenever $N, M \trianglelefteq G$ and $G/N, G/M \in \mathfrak{F}$, and (2) \mathfrak{F} is closed under taking homomorphic images, that is, $G/N \in \mathfrak{F}$ whenever $N \trianglelefteq G$ and $G \in \mathfrak{F}$;

(ii) *hereditary* (Mal'cev [8]) if $G \in \mathfrak{F}$ whenever $G \leq A \in \mathfrak{F}$;

(iii) *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$;

(iv) a *Fitting class* if the following two conditions hold: (1) \mathfrak{F} is normally hereditary, that is, $G \in \mathfrak{F}$ whenever $G \trianglelefteq A \in \mathfrak{F}$, and (2) \mathfrak{F} is closed under taking products of normal subgroups, that is, $G \in \mathfrak{F}$ whenever $G = AB$, where $A, B \trianglelefteq G$ and $A, B \in \mathfrak{F}$.

If $K \leq H \leq G$, where $K, H \trianglelefteq G$, then we say that H/K is a *normal factor* or a *normal section* of G .

Definition 1.7. (i) If G is σ_3 -nilpotent, where

$$\sigma = \{\sigma_0, \sigma_1\} = \sigma^\pi$$

and $\sigma_0 = \pi'$, $\sigma_1 = \pi$, then we say that G is (π, \mathfrak{S}) -decomposable.

(ii) If G is σ_3 -nilpotent, where

$$\sigma = \{\sigma_0, \{p_1\}, \dots, \{p_n\}, \dots\} = \sigma^{1, \pi'}$$

and $\sigma_0 = \pi'$, $\pi = \{p_1, \dots, p_n, \dots\}$ then we say that G is $(1, \mathfrak{S})$ -special.

Definition 1.8. (i) If a subgroup A of G is σ_3 -subnormal in G , where $\sigma = \{\sigma_0, \sigma_1\} = \sigma^\pi$ and $\sigma_0 = \pi'$, $\sigma_1 = \pi$, then we say that A is (π, \mathfrak{S}) -subnormal in G .

(ii) If a subgroup A of G is σ_3 -subnormal in G , where

$$\sigma = \{\sigma_0, \{p_1\}, \dots, \{p_n\}, \dots\} = \sigma^{1, \pi'}$$

and

$$\sigma_0 = \pi', \quad \pi = \{p_1, \dots, p_n, \dots\},$$

then we say that A is $(1, \mathfrak{S})$ -subnormal in G .

Definition 1.9. Let H/K be a normal factor of G . Then we say that H/K is:

(i) σ_3 -central in G if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is σ_3 -primary;

(ii) (π, \mathfrak{S}) -central in G if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is either an \mathfrak{S} -group or is a π -group;

(iii) $(1, \mathfrak{S})$ -central in G if the semidirect product

$$(H/K) \rtimes (G/C_G(H/K))$$

is either an \mathfrak{S} -group or a p -group for some prime $p \in \pi$.

Applications of the σ_3 -nilpotent groups and σ_3 -subnormal subgroups are based on our three results as follows.

Theorem 1.10. (i) The class \mathfrak{N}_{σ_3} , of all σ_3 -nilpotent groups, is both a hereditary saturated formation and a Fitting class.

(ii) If G is σ_3 -nilpotent, then every subgroup of G is σ_3 -subnormal in G .

Theorem 1.11. Any two of the following conditions are equivalent:

(i) G is σ_3 -nilpotent.

(ii) Every subgroup of G is σ_3 -subnormal in G .

(iii) Every maximal subgroup of G is σ_3 -subnormal in G .

(iv) G has a σ_3 -subnormal Hall σ_i -subgroup for every $\sigma_i \in \sigma(G)$.

(v) Every chief factor of G is σ_3 -central in G .

(vi) every factor of some normal series of G is σ_3 -central in G .

In view of Example 1.6 (iii) (iv) we get from Theorem 1.11 the following results.

Corollary 1.12. A group G is (π, \mathfrak{S}) -decomposable if and only if every subgroup of G is (π, \mathfrak{S}) -subnormal in G .

Corollary 1.13 (Kegel [4]). *A group G is $(1, \mathfrak{S})$ -special if and only if every subgroup of G is $(1, \mathfrak{S})$ -subnormal in G .*

In the case when \mathfrak{S} is the class of all σ_0 -groups we get from Corollaries 1.12 and 1.13 the following new results.

Corollary 1.14. *A group G is π -decomposable if and only if for every subgroup A of G there is a subgroup chain*

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is either a π -group or a π' -group.

Corollary 1.15. *A group G is π -special if and only if for every subgroup A of G there is a subgroup chain*

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is a π' -group.

We get also from Theorem 1.11 the following results.

Corollary 1.16. *A group G is (π, \mathfrak{S}) -decomposable if and only if every chief factor of G is (π, \mathfrak{S}) -central in G .*

Corollary 1.17. *A group G is (π, \mathfrak{S}) -special if and only if every chief factor of G is $(1, \mathfrak{S})$ -central in G .*

In the case when \mathfrak{S} is the class of all σ_0 -groups we get also from Theorem 1.11 the following known result.

Corollary 1.18 (W. Guo, A.N. Skiba [9]). *Any two of the following conditions are equivalent:*

- (i) G is σ -nilpotent.
- (ii) Every chief factor of G is σ -central in G .
- (iii) G has a σ -subnormal Hall σ_i -subgroup for every $\sigma_i \in \sigma(G)$.

(iv) Every subgroup of G is σ -subnormal in G .

(v) Every maximal subgroup of G is σ -subnormal in G .

We use $O^\Pi(G)$ to denote the subgroup of G generated by all its Π' -subgroups, where $\Pi' = \sigma \setminus \Pi$; $O_\Pi(G)$ is the product of all normal Π -subgroups of G . We also use G^{σ_3} to denote the σ_3 -nilpotent residual of G , that is, the intersection of all normal subgroups N of G with σ_3 -nilpotent G/N .

Theorem 1.19. *Let A, K and N be subgroups of G , where A is σ_3 -subnormal and N is normal in G .*

- (i) $A \cap K$ is σ_3 -subnormal in K .
- (ii) If K is a σ_3 -subnormal subgroup of A , then K is σ_3 -subnormal in G .
- (iii) AN/N is σ_3 -subnormal in G/N .

(iv) If $N \leq K$ and K/N is σ_3 -subnormal in G/N , then K is σ_3 -subnormal in G .

(v) If $K \leq A$ and A is σ_3 -primary, then K is σ_3 -subnormal in G .

(vi) The σ_3 -nilpotent residual A^{σ_3} of A is subnormal in G .

(vii) If A is σ_3 -nilpotent, then A^G is σ_3 -nilpotent and $\sigma(A) = \sigma(A^G)$.

(viii) If $\sigma(|G:A|) \subseteq \Pi$, then $O^\Pi(A) = O^\Pi(G)$.

(ix) If A is a σ -Hall subgroup of G , then A is normal in G .

2 Proofs of the results

Proof of Theorem 1.4. First assume that A is K - \mathfrak{S} -subnormal in G and

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G,$$

where either $A_{i-1} \trianglelefteq A_i$ or

$$A_i / (A_{i-1})_{A_i} \in \mathfrak{S}$$

for all $i = 1, \dots, t$, then either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ_3 -primary. Hence A is σ_3 -subnormal in G .

Now assume that A is σ_3 -subnormal in G and let

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G,$$

where either $A_{i-1} \trianglelefteq A_i$, or $A_i / (A_{i-1})_{A_i} \in \mathfrak{S}$, or

$$A_i / (A_{i-1})_{A_i}$$

is a p -group for some prime p . But if $A_i / (A_{i-1})_{A_i}$ is a p -group, then A_{i-1} is subnormal in A_i . Therefore there is a subgroup chain

$$A = B_0 \leq B_1 \leq \dots \leq B_n = G$$

such that either $B_{i-1} \trianglelefteq B_i$ or

$$B_i / (B_{i-1})_{B_i} \in \mathfrak{S}$$

for all $i = 1, \dots, n$, so A is K - \mathfrak{S} -subnormal in G . \square

Proof of Theorem 1.5. First suppose that A is $\Pi_{\mathfrak{S}}$ -subnormal in G and let

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G,$$

where $A_{i-1} \trianglelefteq A_i$, or

$$A_i / (A_{i-1})_{A_i} \in \mathfrak{S},$$

or $A_i / (A_{i-1})_{A_i}$ is a σ_j -group for some

$$\sigma_j \in \Pi \setminus \{\sigma_0\}.$$

But in the last two cases, $A_i / (A_{i-1})_{A_i}$ is σ_3 -primary and so A is σ_3 -subnormal in G .

Finally, if A is σ_3 -subnormal in G and

$$A = A_0 \leq A_1 \leq \dots \leq A_t = G,$$

where $A_i / (A_{i-1})_{A_i}$ is σ_3 -primary for some i , then $A_i / (A_{i-1})_{A_i}$ is Π_3 -primary (see Part (i)) and so every σ_3 -subnormal is also Π_3 -subnormal in G . \square

Lemma 2.1. Let $G \neq 1$ and $\sigma(G) = \{\sigma_{i_1}, \dots, \sigma_{i_t}\}$.

Then G is σ_3 -nilpotent if and only if

$$G = H_1 \times \dots \times H_t,$$

where H_j is a Hall σ_{i_j} -subgroup of G for all $j = 1, \dots, t$ and $H_k \in \mathfrak{S}$ for $i_k = 0$.

Proof. Indeed, assume that $G = A_1 \times \dots \times A_n$, where A_i is σ_3 -primary for all $i = 1, \dots, n$, and let H_j be the product of all factors A_i such that A_i is a σ_{i_j} -group. Then H_j is a Hall σ_{i_j} -subgroup of G for all j and, clearly, $H_k \in \mathfrak{S}$ for $i_k = 0$ since the \mathfrak{S} is closed under extensions and epimorphic images. In particular, $G = H_1 \times \dots \times H_t$. \square

Proof of Theorem 1.10. In view of Lemma 2.1, this theorem can be proved by the direct checking.

Proof of Theorem 1.19. Assume that this theorem is false and let G be a counterexample of minimal order. Then $1 \neq A \neq G$ since, in the cases $A = 1$ and $A = G$, the Statements (i)–(ix) trivially hold for G .

By hypothesis, there is a subgroup chain

$$A = A_0 \leq A_1 \leq \dots \leq A_r = G$$

such that either A_{i-1} is normal in A_i or $A_i / (A_{i-1})_{A_i}$ is σ_3 -primary for all $i = 1, \dots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$ since $A \neq G$.

(i) Consider the chain

$$K_0 = K \cap A_0 \leq K \cap A_1 \leq \dots \leq K \cap A_r = K.$$

If A_{i-1} is normal in A_i , then $K \cap A_{i-1}$ is normal in $K \cap A_i$.

Suppose that $A_i / (A_{i-1})_{A_i}$ is an \mathfrak{S} -group. Then $(A_i \cap K) / (A_{i-1})_{A_i} / (A_{i-1})_{A_i} \cong (A_i \cap K) / ((A_{i-1})_{A_i} \cap K)$ is an \mathfrak{S} -group, where $(A_{i-1})_{A_i} \cap K$ is normal in $A_i \cap K$ and so

$$(A_{i-1})_{A_i} \cap K \leq (K \cap A_{i-1})_{K \cap A_i}.$$

Hence $(K \cap A_i) / (K \cap A_{i-1})_{K \cap A_i}$ is an \mathfrak{S} -group. Finally, if $A_i / (A_{i-1})_{A_i}$ is a σ_j -group for some $j \neq 0$, then similarly we get that $(K \cap A_i) / (K \cap A_{i-1})_{K \cap A_i}$ is a σ_j -group. Therefore $(K \cap A_i) / (K \cap A_{i-1})_{K \cap A_i}$ is σ_3 -primary for all $i = 1, \dots, r$, so $A \cap K$ is σ_3 -subnormal in K .

(ii) Assume that K is σ_3 -subnormal in A and let

$$K = K_0 \leq K_1 \leq \dots \leq K_m = A,$$

where either $K_{i-1} \trianglelefteq K_i$ or $K_i / (K_{i-1})_{K_i}$ is σ_3 -primary for all $i = 1, \dots, m$. Then the subgroup series

$$K = K_0 \leq K_1 \leq \dots \leq K_m = A = A_0 \leq A_1 \leq \dots \leq A_r = G$$

has the same property in G and so K is σ_3 -subnormal in G .

(iii) Consider the chain

$$AN / N = A_0N / N \leq A_1N / N \leq \dots \leq A_rN / N = G / N.$$

Assume that $A_{i-1}N / N$ is not normal in A_iN / N . Then $L = A_{i-1}$ is not normal in $T = A_i$ and so T / L_T is σ_3 -primary. Hence

$$\begin{aligned} (T / L_T) / (L_T(T \cap N) / L_T) &= \\ &= (T / L_T) / ((T \cap NL_T) / L_T) \cong \end{aligned}$$

$$T / (T \cap NL_T) \cong TN / L_TN \cong (TN / N) / (L_TN / N)$$

is σ_3 -primary. But

$$L_TN / N \leq (LN / N)_{TN/N}.$$

Hence $(TN / N) / (LN / N)_{TN/N}$ is σ_3 -primary, so AN / N is σ_3 -subnormal in G / N .

(iv) Consider the chain

$$K / N = K_0 / N \leq K_1 / N \leq \dots \leq K_n / N = G / N$$

be a subgroup chain such that either K_{i-1} / N is normal in K_i / N or $(K_i / N) / (K_{i-1} / N)_{K_i / N}$ is σ_3 -primary for all $i = 1, \dots, n$. Suppose that K_{i-1} is not normal in K_i . Then K_{i-1} / N is not normal in K_i / N , so

$$\begin{aligned} (K_i / N) / (K_{i-1} / N)_{K_i / N} &= \\ &= (K_i / N) / ((K_{i-1})_{K_i} / N) \cong K_i / (K_{i-1})_{K_i} \end{aligned}$$

is σ_3 -primary. Hence K is σ_3 -subnormal in G .

(v) Since A is σ_3 -primary, every subgroup of

A is σ_3 -subnormal in A . Thus this assertion is a corollary of Part (ii).

(vi) First we show that $V := A^{\mathfrak{N}_{\sigma_3}} \leq M_G$.

Indeed, if M is normal in G , it is clear. Now assume that G / M_G is an \mathfrak{S} -group. Then from the isomorphism

$$AM_G / M_G \cong A / (A \cap M_G) \in \mathfrak{S} \subseteq \mathfrak{N}_{\sigma_3}$$

it follows that $V \leq A \cap M_G$.

Finally, if G / M_G is a σ_j -group for some $j \neq 0$, then similarly we get that $V \leq M_G$.

The choice of G implies that V is subnormal in M , so A is subnormal in M_G . Therefore V is subnormal in G .

(vii) Let $E = NA$. Then $N = N_1 \times \dots \times N_n$ for some minimal normal subgroups N_1, \dots, N_n of E and N_i is not σ_3 -primary for all i by Lemma 2.1.

Assume that $E < G$. By Part (i), A is σ_3 -subnormal in E , so due to the choice of G this means that $A \leq C_E(N_i)$ for all i . Hence $A \leq C_E(N)$.

Finally, assume that $NA = E = G$. Then $N \not\leq M$ and hence G/M_G is not σ_3 -primary, since $N \approx NM_G/M_G$ is not σ_3 -primary. This implies that M is normal in G and hence $N \cap M = 1$. Therefore $M \leq C_G(N)$ and so $A \leq C_G(N)$.

(viii) Since $A \leq M$, it follows that $|M:A|$ and $|G:M|$ are Π -numbers. Moreover, A is σ_3 -subnormal in M , so the choice of G implies that

$$O^\Pi(A) = O^\Pi(M).$$

Now we show that G/M_G is a Π -number. If M is normal in G , then $M_G = M$ and so $|G:M_G| = |G:M|$ is a Π -number. Now assume that G/M_G is σ_i -group for some $i \in I$. Then $|G/M|$ is a σ_i -number, so $\sigma_i \in \Pi$ since $|G:M|$ is a Π -number. Hence G/M_G is a Π -number.

Now let V be any Π' -subgroup of G . Then from $VM_G/M_G \leq G/M_G$ and

$$VM_G/M_G \approx V/(V \cap M_G)$$

it follows that VM_G/M_G is simultaneously a Π' -group and a Π -group, so VM_G/M_G is the identity group, that is, $V \leq M_G$. Therefore every Π' -subgroup of G is contained in M_G , so

$$O^\Pi(G) \leq O^\Pi(M_G) \leq O^\Pi(M) \leq O^\Pi(G).$$

Hence

$$O^\Pi(G) = O^\Pi(M) = O^\Pi(A).$$

(ix) By hypothesis, A is a Hall Π -subgroup of G for some $\Pi \subseteq \sigma$. Then A^x is a Hall Π -subgroup of G for every $x \in G$, so $A^x \cap A$ is a Hall Π -subgroup of A . Hence $|A:A^x \cap A|$ is a Π' -number and so $A = A^x \cap A$. Therefore $A \leq A^x$, so $A = A^x$ for all $x \in G$. \square

Lemma 2.2 [11, 3.29]. *Let R be an abelian minimal normal subgroup of G such that $G = RM$ for a maximal subgroup M of G . Then*

$$G/M_G \approx R \rtimes (G/C_G(R)).$$

Now, we are in position to prove Theorem 1.11.

Proof of Theorem 1.11. Since Conditions (i) and (iv) are equivalent by Lemma 2.1 and Theorem 1.19 (ix), (ii) \Rightarrow (iii), and Conditions (v) and (vi) are equivalent by the Jordan – Hölder theorem for the chief series. it is enough to prove the implications (i) \Rightarrow (ii), (i) \Rightarrow (vi), (iii) \Rightarrow (i), (v) \Rightarrow (i).

Let $G \neq 1$ and

$$\sigma(G) = \{\sigma_{i_1}, \dots, \sigma_{i_t}\}.$$

(i) \Rightarrow (ii), (i) \Rightarrow (vi).

Assume that G is σ_3 -nilpotent, so $G = H_1 \times \dots \times H_t$, where H_j is a Hall σ_{i_j} -subgroup of G for all $j = 1, \dots, t$ and $H_k \in \mathfrak{S}$ for $i_k = 0$, by Lemma 2.1.

Suppose that some subgroup A of G is not σ_3 -subnormal in G . Then $t > 1$ by Theorem 1.19 (v) and $1 < A < G$, so $A \leq M$ for some maximal subgroup M of G and for some i , for $i = 1$ say, we have $H_1 \not\leq M$.

It follows that

$$H_2 \cdots H_t \leq M,$$

so $M/H_2 \cdots H_t$ is σ_3 -subnormal in $G/H_2 \cdots H_t$ by induction since $G/H_2 \cdots H_t$ is σ_3 -nilpotent by Theorem 1.10 (i). Therefore M is σ_3 -subnormal in G by Theorem 1.19 (iv). In view of Theorem 1.10 (i), M is also σ_3 -nilpotent, so A is σ_3 -subnormal in M by induction on $|G|$. Therefore A is σ_3 -subnormal in G Theorem 1.19 (ii). This contradiction shows that (i) \Rightarrow (ii).

Now, consider the normal series

$$1 \leq H_1 \leq H_1 H_2 \leq \dots \leq H_1 \cdots H_{t-1} \leq H_1 \times \dots \times H_t = G$$

of G . If H/K is a chief factor of G such that

$$H_1 \cdots H_{j-1} \leq K < H \leq H_1 \cdots H_j,$$

then H/K is a σ_{i_j} -group since

$$H_1 \cdots H_i / H_1 \cdots H_{j-1} \approx H_j,$$

and

$$H_1 \cdots H_{j-1} \leq C_G(H/K).$$

On the other hand, from

$$G = (H_1 \cdots H_j) \times H_{j+1} \cdots H_t$$

it follows that

$$H_{j+1} \cdots H_t \leq C_G(H/K),$$

so $G/C_G(H/K)$ is a σ_{i_j} -group.

Moreover, if $i_j = 0$, then H/K and $G/C_G(H/K)$ belong to \mathfrak{S} , hence

$$(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{S}.$$

Hence G has a chief series whose all factors are σ_3 -central in G , so Statement (vi) holds for G .

Now, we show that if every maximal subgroup of G is σ_3 -subnormal in G , then G is σ_3 -nilpotent. Assume that this is false. Then G is not σ_3 -primary and, by Theorem 1.19 (iii), the hypothesis holds for G/R , so G/R is σ_3 -nilpotent by induction on $|G|$. Therefore R is a unique minimal normal subgroup of G and $R \not\leq \Phi(G)$ by Theorem 1.10 (i). Hence for some maximal subgroup M of G we have $G = RM$ and $M_G = 1$. It is clear that $M \neq 1$, so M

is not normal in G . Therefore $G \cong G/M_G$ is σ_3 -primary, a contradiction. Hence (iii) \Rightarrow (i).

Next we show that if every chief factor of G is σ_3 -central in G , then G is σ_3 -nilpotent. Assume that this is false and let R be a minimal normal subgroup of G . It is clear that the hypothesis holds for G/R , so G/R is σ_3 -nilpotent by induction. Therefore R is a unique minimal normal subgroup of G and $R \not\leq \Phi(G)$ by Theorem 1.10 (i). Hence for some maximal subgroup M of G we have $G = RM$ and $M_G = 1$, so $C_G(R) \leq R$ by Lemma 2.2. If R is non-abelian, $C_G(R) = 1$, hence $G \cong G/C_G(R)$ is σ_3 -primary. Therefore σ_3 -nilpotent, a contradiction. Hence $C_G(R) = R$ is abelian group, so

$$G \cong G/M_G \cong R \rtimes (G/C_G(R))$$

is σ_3 -primary by Lemma 2.2, so G is σ_3 -nilpotent. Thus, (v) \Rightarrow (i). \square

REFERENCES

1. Skiba, A.N. On σ -subnormal and σ -permutable subgroups of finite groups / A.N. Skiba // J. Algebra. – 2015. – Vol. 436. – P. 1–16.
2. Skiba, A.N. On Some Results in the Theory of Finite Partially Soluble Groups / A.N. Skiba // Commun. Math. Stat. – 2016. – Vol. 4. – P. 281–309.
3. Skiba, A.N. Some characterizations of finite σ -soluble $P\sigma T$ -groups / A.N. Skiba // J. Algebra. – 2018. – Vol. 495. – P. 114–129.
4. Kegel, O.H. Untergruppenverbände endlicher Gruppen, die den subnormalteilerverband each enthalten / O.H. Kegel // Arch. Math. – 1978. – Vol. 30, № 33. – S. 225–228.
5. Ballester-Bolinches, A. Classes of Finite Groups / A. Ballester-Bolinches, L.M. Ezquerro. – Springer Verlag, Dordrecht, 2006.
6. Guo, W. On the lattice of Π_3 -subnormal subgroups of a finite group / W. Guo, A.N. Skiba // Bull. Aust. Math. Soc. – 2017. – Vol. 96. – P. 233–244.
7. Chunikhin, S.A. Subgroups of finite groups / S.A. Chunikhin. – Minsk: Nauka i Tehnika, 1964.
8. Mal'cev, A.I. Algebraic Systems / A.I. Mal'cev. – Moscow: Nauka, 1970.
9. Guo, W. Finite groups whose N -maximal subgroups are σ -subnormal / W. Guo, A.N. Skiba // Science in China. Math. – 2019. – Vol. 62, № 7. – P. 1355–1372.
10. Huppert, B. Endliche Gruppen I / B. Huppert. – Springer Verlag, Berlin – Heidelberg – New York, 1967.
11. Шеметков, Л.А. Формации алгебраических систем / Л.А. Шеметков, А.Н. Скиба. – Москва: Наука, 1989.
12. Doerk, K. Finite Soluble Groups / K. Doerk, T. Hawkes. – Walter de Gruyter, Berlin – New York, 1992.

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