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КЛАССИЧЕСКОЕ И СЛАБОЕ РЕШЕНИЕ ПЕРВОЙ СМЕШАННОЙ ЗАДАЧИ ДЛЯ ПОЛУЛИНЕЙНОГО ВОЛНОВОГО УРАВНЕНИЯ СО СМЕШАННОЙ ПРОИЗВОДНОЙ

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CLASSICAL AND MILD SOLUTIONS OF THE FIRST MIXED PROBLEM FOR A SEMILINEAR WAVE EQUATION WITH A MIXED DERIVATIVE

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Аннотация. Для полулинейного волнового уравнения со смешанной производной, заданного в первом квадранте, рассматривается смешанная задача, в которой на пространственной полупрямой заданы условия Коши, а на временной – условия Дирихле. Оператор в уравнении представляет собой композицию двух операторов переноса с постоянными коэффициентами. Решение строится методом характеристик в неявном аналитическом виде как решение некоторых интегральных уравнений. Исследуется разрешимость этих уравнений, а также зависимость от начальных данных и гладкость их решений. Для рассматриваемой задачи доказана единственность решения и установлены условия существования ее классического решения. Построено слабое решение в случае недостаточно гладких данных задачи.

Ключевые слова: *гиперболическое уравнение*, *волновое уравнение со смешанной производной*, *нелинейное волновое уравнение*, *смешанная задача*, *классическое решение*, *слабое решение*, *условия согласования.*

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Abstract. We consider a mixed problem for a semilinear wave equation with a mixed derivative in the first quadrant, where the Cauchy conditions are specified on the spatial half-line and the Dirichlet condition is specified on the time half-line. The operator in the equation is a composition of two transport operators with constant coefficients. We construct the solution using the method of characteristics as an implicit analytical form that solves some integral equations. We study the solvability of these equations, as well as their dependence on the source data and the smoothness of the solutions. The uniqueness of the solution is proved for the problem in question, and the conditions under which its classical solution exists are established. In the case of insufficiently smooth data, a mild solution is constructed.

Keywords: *hyperbolic equation*, *wave equation with mixed derivative*, *nonlinear wave equation*, *mixed problem*, *classical solution*, *mild solution*, *matching conditions.*

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Introduction

Nonlinear wave equations of the form

$$(\partial_t^2 - a^2\partial_x^2)u(t,x) = f(t,x,u(t,x))$$

appear in many applications, including the propagation of spin waves in liquid ^{3}He, the propagation of resonant light pulses in a medium with atoms having degenerate energy levels [1], the quantum field theory [2, 3], rapidly rotating baroclinic fluid [4], superconductors [4], Bloch-wall motion, the propagation of a crystal dislocation [5]. It demonstrates the importance of studying mixed problems for the nonlinear wave equations. The simplest generalization of Eq. $(\partial_t^2 - a^2\partial_x^2)u(t,x) = f(t,x,u(t,x))$ is the following equation

$$(\partial_t + a_1\partial_x)(\partial_t - a_2\partial_x)u(t,x) = f(t,x,u(t,x)),$$

which will be studied in the present paper.

1 Statement of the problem

In the domain $Q = (0,\infty)\times(0,\infty)$ of two independent variables $(t,x)\in\overline{Q}\subset\mathbb{R}^2$, consider the one-dimensional nonlinear equation

$$(\partial_t + a_1\partial_x)(\partial_t - a_2\partial_x)u(t,x) = f(t,x,u(t,x)), \quad (1.1)$$

where $a_1 > 0$, $a_2 > 0$, and f is a function given on the set $[0,\infty)\times[0,\infty)\times\mathbb{R}$. Equation (1.1) is equipped with the initial conditions

$$u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x),\ x\in[0,\infty),\ (1.2)$$

and the boundary condition

$$u(t,0)=\mu(t),\ t\in[0,\infty),\qquad (1.3)$$

where φ, ψ, and μ are functions given on the half-line [0, ∞).

We call Eq. (1.1) a semilinear wave equation with a mixed derivative, because its principal part is of the form:

$$(\partial_t+a_1\partial_x)(\partial_t-a_2\partial_x)=\partial_t^2+(a_1-a_2)\partial_t\partial_x-a_1a_2\partial_x^2,$$

where terms ∂_t^2 and $-a_1a_2\partial_x^2$ correspond to the terms ∂_t^2 and $-a^2\partial_x^2$ of the classical wave equation, respectively, and the term $(a_1-a_2)\partial_t\partial_x$ corresponds to an additional mixed derivative.

In the case $a_1=a_2$ problem (1.1)–(1.3) was considered in [6]–[9]. Mixed problems for Eq. (1) in the linear case, i. e., $f(t,x,u)=F(t,x)+q(t,x)u$, were considered in [10]–[13].

2 Linear homogeneous equation

First, we consider problem (1.1)–(1.3) when

$$f(t,x,u)\equiv 0.\qquad (2.1)$$

We find the solution to problem (1.1)–(2.1) using the method of characteristics. The general solution to Eq. (1.1) with (2.1) is given by the formula:

$$u(t,x)=g_1(x-a_1t)+g_2(x+a_2t).\qquad (2.2)$$

To define the solution of problem (1.1)–(2.1) on the set $\overline{Q}$, the functions g_1 and g_2 must be defined for all real numbers and nonnegative real numbers, respectively. Substituting (2.2) into the Cauchy conditions (1.2) yields the following system of equations:

$$g_1(x)+g_2(x)=\varphi(x),$$
$$-a_1g_1(x)+a_2g_2(x)=\psi'(x),\ x\in[0,\infty),$$

which has the following solution (see source [14] for an algorithm to solve this system):

$$g_1(x)=\frac{a_2\varphi(x)}{a_1+a_2}-\frac{C}{a_1+a_2}-\frac{1}{a_1+a_2}\int_0^x\psi(z)dz,\ (2.3)$$
$$x\in[0,\infty),$$

$$g_2(x)=\frac{a_1\varphi(x)}{a_1+a_2}+\frac{C}{a_1+a_2}+\frac{1}{a_1+a_2}\int_0^x\psi(z)dz,\ (2.4)$$
$$x\in[0,\infty),$$

where C is an integration constant.

We can use the boundary conditions (1.3) to define the function g_1 for negative values of the argument. Substituting (2.2) into the boundary conditions (1.3) yields the following representation:

$$g_1(-a_1t)+g_2(a_2t)=\mu(t),$$

which implies

$$g_1(x)=-g_2\left(-\frac{a_2x}{a_1}\right)+\mu\left(-\frac{x}{a_1}\right),\ x\in(-\infty,0].\ (2.5)$$

Combining (2.5) with (2.3), we get

$$g_1(x)=\mu\left(-\frac{x}{a_1}\right)-\frac{a_2\varphi\left(-\frac{a_2x}{a_1}\right)}{a_1+a_2}-$$
$$-\frac{C}{a_1+a_2}-\frac{1}{a_1+a_2}\int_0^{-\frac{a_2x}{a_1}}\psi(z)dz,\ x\in(-\infty,0].\ (2.6)$$

Formulas (2.3), (2.4), and (2.6) imply that

$$u(t,x)=\frac{a_2\varphi(x-a_1t)+a_1\varphi(x-a_2t)}{a_1+a_2}+$$
$$+\frac{1}{a_1+a_2}\int_{x-a_1t}^{x+a_2t}\psi(z)dz,\ (t,x)\in\overline{Q^{(1)}},\qquad (2.7)$$

$$u(t,x)=\mu\left(t-\frac{x}{a_1}\right)+\frac{a_1\varphi(x+a_2t)}{a_1+a_2}-\frac{a_1\varphi\left(a_2\left(t-\frac{x}{a_1}\right)\right)}{a_1+a_2}+$$
$$\frac{1}{a_1+a_2}\int_{a_2\left(t-\frac{x}{a_1}\right)}^{x+a_2t}\psi(z)dz,\ (t,x)\in\overline{Q^{(2)}},\qquad (2.8)$$

where $Q^{(j)}=\{(t,x):(t,x)\in Q\wedge(-1)^j(a_1t-x)>0\}$, $j=1,2$.

Now, let us analyze the smoothness of the constructed solution. Under the smoothness conditions $\varphi\in C^2([0,\infty))$, $\psi\in C^1([0,\infty))$, and $\mu\in C^2([0,\infty))$ the function u defined by formulas (2.7) and (2.8) belongs to the classes $C^2(\overline{Q^{(1)}})$ and $C^2(\overline{Q^{(2)}})$. Thus, this function u belongs to the class $C^2(\overline{Q})$ if and only if the following conditions are met:

$$\partial_t^p\partial_x^k u(t,x=at-0)=\partial_t^p\partial_x^k u(t,x=at+0),\ (2.9)$$
$$p,k=0,1,2,\ 0\le p+k\le 2.$$

Substituting (2.7) and (2.8) into conditions (2.9), we obtain the following equalities:

$$\mu(0)=\varphi(0),\qquad (2.10)$$

$$\mu'(0)=\psi(0),\qquad (2.11)$$

$$\mu''(0)=-a_1\psi'(0)+a_2\psi'(0)+a_1a_2\varphi''(0).\ (2.12)$$

Thus, we have proved the following assertion.

***Assertion* 2.1.** *Let the conditions be satisfied*

$$\varphi\in C^2([0,\infty)),\ \psi\in C^1([0,\infty)),\ \mu\in C^2([0,\infty)).$$

The first mixed problem (1.1)–(2.1) *has a unique solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C^2(\overline{Q})$ *if and only if conditions* (2.10)–(2.12) *are satisfied. This solution is determined by formulas* (2.7) *and* (2.8).

Proof. 1. The *existence* of the solution follows from the construction. 2. The *uniqueness* of the solution follows from the rigorous justification of each step in the construction process. □

3 Linear inhomogeneous equation

Now consider Eq. (1.1) with

$$f(t,x,u)=F(t,x),\qquad (3.1)$$

where F is some function given on the set $\overline{Q}$.

According to Duhamel's method [14], we seek a solution to problem (1.1)–(1.3) and (3.1) in the form $u_{\text{inhom}} = u + v$, where u is a solution to problem (1.1)–(1.3) and (3.1) for $F \equiv 0$ and v is determined in terms of the function

$$\omega : [0,\infty)\times[0,\infty)\times[0,\infty) \ni (t,\tau,x) \mapsto \omega(t,\tau,x) \in \mathbb{R}$$

by the formula

$$v(t,x) = \int_0^t \omega(t-\tau,\tau,x)d\tau. \quad (3.2)$$

where the function ω solves the following problem:

$$(\partial_t + a_1\partial_x)(\partial_t - a_2\partial_x)\omega(t,\tau,x) = 0,\ (t,x)\in Q,$$
$$\omega(0,\tau,x) = 0,\ \partial_t\omega(0,\tau,x) = F(\tau,x),\ x\in[0,\infty),$$
$$\omega(t,\tau,0) = 0,\ t\in[0,\infty).$$

According to Section 2, the function ω has the following form:

$$\omega(t,\tau,x) = \frac{1}{a_1+a_2}\int_{x-a_1 t}^{x+a_2 t} F(\tau,z)dz,\ (t,x)\in\overline{Q^{(1)}}, \quad (3.3)$$

$$\omega(t,\tau,x) = \frac{1}{a_1+a_2}\int_{a_2\left(t-\frac{x}{a_1}\right)}^{x+a_2 t} F(\tau,z)dz,\ (t,x)\in\overline{Q^{(2)}}. \quad (3.4)$$

So, according to formulas (3.2)–(3.4), the function v takes the form

$$v(t,x) = \frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F(\tau,z)dz,\ (t,x)\in\overline{Q^{(2)}},$$

$$v(t,x) = \frac{1}{a_1+a_2}\int_{t-\frac{x}{a_1}}^{t} d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F(\tau,z)dz +$$
$$+\frac{1}{a_1+a_2}\int_0^{t-\frac{x}{a_1}} d\tau \int_{a_2\left(t-\frac{x}{a_1}-\tau\right)}^{x+a_2(t-\tau)} F(\tau,z)dz,\ (t,x)\in\overline{Q^{(2)}}.$$

Thus, we have constructed the solution to problem (1.1)–(1.3) and (3.1) in the form

$$u(t,x) = \frac{a_2\varphi(x-a_1t) + a_1\varphi(x-a_2t)}{a_1+a_2} +$$
$$+\frac{1}{a_1+a_2}\int_{x-a_1t}^{x+a_2t}\psi(z)dz +$$
$$+\frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F(\tau,z)dz,\ (t,x)\in\overline{Q^{(1)}}, \quad (3.5)$$

$$u(t,x) = \mu\left(t-\frac{x}{a_1}\right) + \frac{a_1\varphi(x+a_2t)}{a_1+a_2} -$$
$$-\frac{a_1\varphi\left(a_2\left(t-\frac{x}{a_1}\right)\right)}{a_1+a_2} + \frac{1}{a_1+a_2}\int_{a_2\left(t-\frac{x}{a_1}\right)}^{x+a_2t}\psi(z)dz +$$
$$+\frac{1}{a_1+a_2}\int_{t-\frac{x}{a_1}}^{t} d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F(\tau,z)dz +$$
$$+\frac{1}{a_1+a_2}\int_0^{t-\frac{x}{a_1}} d\tau \int_{a_2\left(t-\frac{x}{a_1}-\tau\right)}^{x+a_2(t-\tau)} F(\tau,z)dz,\ (t,x)\in\overline{Q^{(2)}}. \quad (3.6)$$

Now, let us analyze the smoothness of the constructed solution. Under the smoothness conditions $F\in C^1(\overline{Q})$, $\varphi\in C^2([0,\infty))$, $\psi\in C^1([0,\infty))$, and $\mu\in C^2([0,\infty))$ the function u defined by formulas (3.5) and (3.6) belongs to the classes $C^2(\overline{Q^{(1)}})$ and $C^2(\overline{Q^{(2)}})$. Thus, this function u belongs to the class $C^2(\overline{Q})$ if and only if conditions (2.9) are met. Substituting (3.5) and (3.6) into conditions (2.9) yields (2.10), (2.11), and

$$\mu''(0) = F(0,0) - a_1\psi'(0) + a_2\psi'(0) + a_1a_2\varphi''(0). \quad (3.7)$$

The following assertion holds.

***Assertion* 3.1.** *Let the conditions be satisfied*

$$F\in C^1(\overline{Q}),\ \varphi\in C^2([0,\infty)),\ \psi\in C^1([0,\infty)),$$
$$\mu\in C^2([0,\infty)).$$

The first mixed problem (1.1)–(1.3) *and* (3.1) *has a unique solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C^2(\overline{Q})$ *if and only if conditions* (2.10), (2.11), *and* (3.7) *are satisfied. This solution is determined by formulas* (3.5) *and* (3.6).

Proof. 1. The *existence* of the solution follows from the construction. 2. Let us prove the *uniqueness*. Assume that problem (1.1)–(1.3) and (3.1) has two solutions, namely, $u^{(1)}$ and $u^{(2)}$. Denote $U = u^{(1)} - u^{(2)}$. Then the function U solves the following problem

$$(\partial_t + a_1\partial_x)(\partial_t - a_2\partial_x)U(t,x) = 0,\ (t,x)\in Q,$$
$$U(t,x) = \partial_t U(t,x) = 0,\ x\in[0,\infty),$$
$$U(t,0) = 0,\ t\in[0,\infty). \quad (3.8)$$

According to Assertion 2.1, problem (3.8) has a unique solution $U\equiv 0$. Thus, $u^{(1)}\equiv u^{(2)}$ and the uniqueness is proved.

4 Integral equation

Under certain smoothness and matching conditions, it turns out that the classical solution to problem (1.1)–(1.3) is equivalent to the solution to two coupled integral equations. The following theorem addresses this equivalence.

***Theorem* 4.1.** *Let the conditions be satisfied*

$$f\in C^1(\overline{Q}\times\mathbb{R}),\ \varphi\in C^2([0,\infty)),$$
$$\psi\in C^1([0,\infty)),\ \mu\in C^2([0,\infty)).$$

The function $u:\overline{Q}\mapsto\mathbb{R}$ *belongs to the class* $C^2(\overline{Q})$ *and satisfies Eq.* (1.1) *with conditions* (1.2) *and* (1.3) *if and only if it is a continuous solution of the following coupled integral equations*

$$u(t,x) = \frac{a_2\varphi(x-a_1t)+a_1\varphi(x-a_2t)}{a_1+a_2} + \frac{1}{a_1+a_2}\int_{x-a_1t}^{x+a_2t}\psi(z)dz + \frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F\left(\tau,z,u(\tau,z)\right)dz, \quad (4.1)$$

$$(t,x)\in\overline{Q^{(1)}},$$

$$u(t,x) = \mu\left(t-\frac{x}{a_1}\right) + \frac{a_1\varphi(x+a_2t)}{a_1+a_2} - \frac{a_1\varphi\left(a_2\left(t-\frac{x}{a_1}\right)\right)}{a_1+a_2} + \frac{1}{a_1+a_2}\int_{a_2\left(t-\frac{x}{a_1}\right)}^{x+a_2t}\psi(z)dz + \frac{1}{a_1+a_2}\int_{t-\frac{x}{a_1}}^{t} d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F\left(\tau,z,u(\tau,z)\right)dz + \frac{1}{a_1+a_2}\int_0^{t-\frac{x}{a_1}} d\tau \int_{a_2\left(t-\frac{x}{a_1}-\tau\right)}^{x+a_2(t-\tau)} F\left(\tau,z,u(\tau,z)\right)dz, \quad (4.2)$$

$$(t,x)\in\overline{Q^{(2)}},$$

and the matching conditions (2.10), (2.11), *and*

$$\mu''(0) = f(0,0,\varphi(0)) - a_1\psi'(0) + a_2\psi'(0) + a_1a_2\varphi''(0) \quad (4.3)$$

are satisfied.

Proof. 1. Let a function $u\in C^2(\overline{Q})$ satisfy Eq. (1.1) and conditions (1.2), (1.3).

We will use the scheme proposed in [8] to derive the matching conditions (2.10), (2.11), and (4.3). By differentiating the initial and boundary conditions, we find that

$$\partial_x u(0,x)=\varphi'(x),\ \ \partial_x^2 u(0,x)=\varphi''(x),\ \ \partial_t\partial_x u(0,x)=\psi'(x),\ \ x\in[0,\infty), \quad (4.4)$$

$$\partial_t u(t,0)=\mu'(t),\ \ \partial_t^2 u(t,0)=\mu''(t),\ \ t\in[0,\infty). \quad (4.5)$$

Thus, we have the following:

$$u(0,0)=\mu(0),\ \ \partial_t u(0,0)=\mu'(0),\ \ \partial_t^2 u(0,0)=\mu''(0), \quad (4.6)$$

$$u(0,0)=\varphi(x),\ \ \partial_t u(0,0)=\psi(x). \quad (4.7)$$

Formulas (4.6) and (4.7) imply conditions (2.10) and (2.11). From Eq. (1.1) we find the quantity $\partial_t^2 u(t,0)$, namely,

$$\partial_t^2 u(t,0) = f(0,0,u(0,0)) - a_1\partial_t\partial_x u(0,0) + a_2\partial_t\partial_x u(0,0) + a_1a_2\partial_x^2 u(0,0). \quad (4.8)$$

Substituting (4.4), (4.6), and (4.7) into (4.8) yields condition (4.3).

We seek the expression for a solution to problem (1.1)–(1.3) in the form $u=v+w$, where v is a solution to the problem

$$(\partial_t+a_1\partial_x)(\partial_t-a_2\partial_x)v(t,x)=0,\ \ (t,x)\in Q,$$
$$v(0,x)=\varphi(x),\ \ \partial_t v(0,x)=\psi(x),\ \ x\in[0,\infty),$$
$$v(t,0)=\mu(t),\ \ t\in[0,\infty),$$

and w is a solution to the problem

$$(\partial_t+a_1\partial_x)(\partial_t-a_2\partial_x)w(t,x)=f\left(t,x,(v+w)(t,x)\right),\ \ (t,x)\in Q,$$
$$w(t,x)=\partial_t w(t,x)=0,\ \ x\in[0,\infty),$$
$$w(t,0)=0,\ \ t\in[0,\infty).$$

Then, the results of Assertion 3.1 allow us to write formulas (4.1) and (4.2).

2. Suppose that the function u can be represented as (4.1)–(4.2) and the conditions (2.10), (2.11), and (4.3) are satisfied. Due to the smoothness conditions $f\in C^1(\overline{Q}\times\mathbb{R})$, $\varphi\in C^2([0,\infty))$, $\psi\in C^1([0,\infty))$, and $\mu\in C^2([0,\infty))$, similarly to [6], we conclude that $u\in C^2(\overline{Q^{(1)}})$ and $u\in C^2(\overline{Q^{(2)}})$. Substituting (4.1)–(4.2) into Eq. (1.1) and conditions (1.2), (1.3), we verify that the function u satisfies Eq. (1.1) in each of the subdomains $\overline{Q^{(1)}}$ and $\overline{Q^{(2)}}$, and the initial and boundary conditions everywhere. In order for the function u to belong to the class $C^2(\overline{Q})$, it is necessary and sufficient that equalities (2.9) be satisfied. For this, it suffices to satisfy conditions (2.10), (2.11), and (4.3), as can be deduced from the representations (4.1) and (4.2), and to act exactly according to the algorithm described in [6], [8]. The proof of the theorem is complete.

***Theorem* 4.2.** *Let the conditions be satisfied*

$$f\in C(\overline{Q}\times\mathbb{R}),\ \ \varphi\in C([0,\infty)),\ \ \psi\in L^1_{\text{loc}}([0,\infty)),\ \ \mu\in C([0,\infty)),$$

and let the function f satisfy the Lipschitz condition with a function $L\in L^1_{\text{loc}}(\overline{Q})$ *with respect to the third variable, i. e.,* $|f(t,x,z)-f(t,x,w)|\le L(t,x)|z-w|$. *Then there exist unique solutions of Eqs.* (4.1) *and* (4.2) *in the classes* $C(\overline{Q^{(1)}})$ *and* $C(\overline{Q^{(2)}})$, *respectively, and these solutions continuously depend on the source data.*

Proof. The proof of the theorem will be carried out by the scheme set forth in [6], [8], [15] (in complete form) and in [14], [16] (briefly).

To be definite, consider Eq. (4.1). It will be solved by the successive approximation method. Set

$$G(t,x) = \frac{a_2\varphi(x-a_1t)+a_1\varphi(x-a_2t)}{a_1+a_2} + \frac{1}{a_1+a_2}\int_{x-a_1t}^{x+a_2t}\psi(z)dz.$$

Take the initial approximation $u^{(0)}(t,x)=G(t,x)$,

$(t,x)\in\overline{Q^{(1)}}$. Then every subsequent approximation will be calculated by the formula

$$u^{(m)}(t,x)=G(t,x)+$$
$$+\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}F\left(\tau,z,u^{(m-1)}(\tau,z)\right)dz,\quad(4.9)$$
$$(t,x)\in\overline{Q^{(1)}},\quad m=1,2,\ldots$$

Let us establish the estimates for the successive approximations. Let

$$\Omega_m=\operatorname{Conv}\left\{(0,0),(0,m),\left(\frac{m}{a_1+a_2},\frac{a_1m}{a_1+a_2}\right)\right\},$$
$$\Lambda_m=\|L\|_{L^2(\Omega_m)},\quad m=1,2,\ldots$$

Then $\left|u^{(1)}(t,x)-u^{(0)}(t,x)\right|\le\mathcal{M}_m$,

$$\left|u^{(2)}(t,x)-u^{(1)}(t,x)\right|\le$$
$$\le\left|\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}F\left(\tau,z,u^{(1)}(\tau,z)\right)dz-\right.$$
$$\left.-\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}F\left(\tau,z,u^{(0)}(\tau,z)\right)dz\right|\le$$
$$\le\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}\left|F\left(\tau,z,u^{(1)}(\tau,z)\right)-\right.$$
$$\left.-F\left(\tau,z,u^{(0)}(\tau,z)\right)\right|dz\le$$
$$\le\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}L(\tau,z)\left|u^{(1)}(\tau,z)-u^{(0)}(\tau,z)\right|dz\le$$
$$\le\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}L(\tau,z)\mathcal{M}dz\le$$
$$\le\frac{1}{a_1+a_2}\sqrt{\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}L^2(\tau,z)dz}\sqrt{\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}\mathcal{M}_m^2dz}\le$$
$$\le\frac{\Lambda_m\mathcal{M}_m t}{\sqrt{2}\sqrt{a_1+a_2}},\quad(t,x)\in\Omega_m,\quad m=1,2,\ldots$$

In what follows, by induction in which the last inequality is chosen as the base case, one can readily prove the estimate

$$\left|u^{(i+1)}(t,x)-u^{(i)}(t,x)\right|\le\frac{\Lambda_m^i\mathcal{M}_m t^i}{\sqrt{(2i)!}\left(a_1+a_2\right)^{i/2}},\quad(4.10)$$
$$(t,x)\in\Omega_m,\quad m=1,2,\ldots$$

Note that

$$u^{(m)}(t,x)=u^{(0)}(t,x)+\sum_{i=0}^{m-1}\left(u^{(i+1)}(t,x)-u^{(i)}(t,x)\right).$$

The estimate (4.10) implies the absolute and uniform convergence of the series

$$u^{(\infty)}(t,x)=u^{(0)}(t,x)+\sum_{i=0}^{\infty}\left(u^{(i+1)}(t,x)-u^{(i)}(t,x)\right),$$

on the set Ω_m for any $m>0$, since its terms are majorized in magnitude by the terms of the uniformly converging series

$$\left\|u^{(0)}\right\|_{C(\Omega_m)}+\sum_{i=0}^{\infty}\frac{\Lambda_m^i\mathcal{M}_m t^i}{\sqrt{(2i)!}\left(a_1+a_2\right)^{i/2}}.$$

Thus, the successive approximations by the continuous functions $u^{(m)}$ uniformly tend on the set Ω_m to a function $u^{(\infty)}:\Omega_m\mapsto\mathbb{R}$ continuous in Ω_m, and, by virtue of arbitrariness of m and the fact that $\bigcup_{m=1}^{\infty}\Omega_m=\overline{Q^{(1)}}$, to a function $u^{(\infty)}:\overline{Q^{(1)}}\mapsto\mathbb{R}$, continuous in $\overline{Q^{(1)}}$. Passing to the limit as $m\to\infty$ in (4.10), we conclude that the function $u^{(\infty)}$ is a solution of Eq. (4.1) on the set $\overline{Q^{(1)}}$.

Let us prove the uniqueness of the solution of Eq. (4.1) by contradiction. Let Eq. (4.1) have two solutions, u_1 and u_2. Denote $U=u_1-u_2$.

Then

$$U(t,x)=\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}F\left(\tau,z,u_1(\tau,z)\right)dz-$$
$$-\frac{1}{a_1+a_2}\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}F\left(\tau,z,u_2(\tau,z)\right)dz,\quad(4.11)$$
$$(t,x)\in\overline{Q^{(1)}}.$$

The function U is continuous, and hence $\left|U(t,x)\right|\le M_{U;m}$ under the condition $(t,x)\in\Omega_m$, where $M_{U;m}$ is some constant. It follows from relation (4.11) in view of the Lipschitz condition and the Cauchy – Bunyakovsky – Schwarz inequality that

$$\left|U(t,x)\right|\le\frac{1}{a_1+a_2}\sqrt{\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}L^2(\tau,z)dz}\times$$
$$\times\sqrt{\int_0^t d\tau\int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)}M_{U;m}^2dz}\le\frac{\Lambda_m M_{U;m}t}{\sqrt{2}\sqrt{a_1+a_2}},$$
$$(t,x)\in\Omega_m,\quad m=1,2,\ldots$$

By induction, we arrive at the estimate

$$\left|U(t,x)\right|\le\frac{\Lambda_m^i\mathcal{M}_m t^i}{\sqrt{(2i)!}\left(a_1+a_2\right)^{i/2}}$$

for any positive integer i and any pair $(t,x)\in\Omega_m$, $m=1,2,\ldots$ It follows that $U\equiv0$ on the set Ω_m, and since m is arbitrary and $\bigcup_{m=1}^{\infty}\Omega_m=\overline{Q^{(1)}}$, we have $U\equiv0$ on the set $\overline{Q^{(1)}}$. This proves the existence of a unique continuous solution of Eq. (4.1).

To prove the continuous dependence of the solution on the initial data, along with Eq. (4.1), consider the perturbed equation

$$(u+\Delta u)(t,x)=(G+\Delta G)(t,x)+$$

$$+\frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F\big(\tau,z,(u+\Delta u)(\tau,z)\big)dz, \quad (4.12)$$

$$(t,x)\in\overline{Q^{(1)}},$$

and the difference of the perturbed equation (4.12) and the unperturbed equation (4.1),

$$\Delta u(t,x)=\Delta G(t,x)+$$

$$+\frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F\big(\tau,z,(u+\Delta u)(\tau,z)\big)dz-$$

$$-\frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} F\big(\tau,z,\Delta u(\tau,z)\big)dz, \quad (4.13)$$

$$(t,x)\in\overline{Q^{(1)}}.$$

For Eq. (4.13) for the disturbance Δu, one has the following estimate of the disturbance modulus:

$$\Delta u(t,x)\le \Delta G(t,x)+$$

$$+\frac{1}{a_1+a_2}\int_0^t d\tau \int_{x-a_1(t-\tau)}^{x+a_2(t-\tau)} L(\tau,z)\Delta u(\tau,z)dz.$$

Applying the multidimensional Grönwall lemma [17, Ch. 13] to the preceding inequality, we obtain $|\Delta u(t,x)|\le C^{(1)}\|G\|_{C(\Omega_m)}$, $(t,x)\in\Omega_m$, $m=1,2,\ldots$, where $C^{(1)}$ is some positive constant depending only on the set Ω_m, the function L, and the numbers a_1 and a_2. It follows from the resulting inequality that however small the perturbation ΔG, $\|G\|_{C(\Omega_m)}=\varepsilon$, is taken, the perturbation of the solution satisfies the inequality $|\Delta u(t,x)|\le\delta=C^{(1)}\varepsilon$ on the set Ω_m, $m=1,2,\ldots$ Since m is arbitrary and $\bigcup_{m=1}^{\infty}\Omega_m=\overline{Q^{(1)}}$, we conclude that the solution u to Eq. (4.1) continuously depends on the source data.

The existence of a unique continuous solution of Eq. (4.2), which continuously depends on the source data, can be proved in a similar way. The proof of the theorem is complete. □

5 Classical solution

The main results of this paper are presented below: the existence and uniqueness of a classical solution to problem (1.1)–(1.3).

***Theorem* 5.1.** *Let the conditions be satisfied*

$$f\in C^1(\overline{Q}\times\mathbb{R}),\ \ \varphi\in C^2([0,\infty)),$$

$$\psi\in C^1([0,\infty)),\ \ \mu\in C^2([0,\infty)),$$

and let the function f satisfy the Lipschitz condition with a function $L\in L^1_{\text{loc}}(\overline{Q})$ *with respect to the third variable, i. e.,* $|f(t,x,z)-f(t,x,w)|\le L(t,x)|z-w|$. *The first mixed problem* (1.1)–(1.3) *has a unique solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C^2(\overline{Q})$ *if and only if conditions* (2.10), (2.11), *and* (4.3) *are satisfied. This solution is determined by formulas* (4.1) *and* (4.2).

The *proof* follows from Theorem 4.1 and 4.2.

6 Mild solution

Now consider problem (1.1)–(1.3) for the case in which the functions μ, φ, ψ, and *f* do not have a sufficient degree of smoothness.

***Definition* 6.1.** We call a function *u* representable in the form (4.1) and (4.2) a *mild solution* of problem (4.1)–(4.3).

***Remark* 6.1.** Any classical solution of problem (4.1)–(4.3) is also a mild solution of this problem.

***Remark* 6.2.** It is also obvious that if the additional smoothness conditions $f\in C^1(\overline{Q}\times\mathbb{R})$, $\varphi\in C^2([0,\infty))$, $\psi\in C^1([0,\infty))$, and $\mu\in C^2([0,\infty))$ are satisfied, as well as the matching conditions (2.10), (2.11) and (4.3), then the mild solution of problem (1.1)–(1.3) is classical.

The following assertion holds.

***Theorem* 6.1.** *Let the conditions be satisfied*

$$f\in C(\overline{Q}\times\mathbb{R}),\ \ \varphi\in C([0,\infty)),$$

$$\psi\in L^1_{\text{loc}}([0,\infty)),\ \ \mu\in C([0,\infty)),$$

and let the function f satisfy the Lipschitz condition with a function $L\in L^1_{\text{loc}}(\overline{Q})$ *with respect to the third variable, i. e.,* $|f(t,x,z)-f(t,x,w)|\le L(t,x)|z-w|$. *The first mixed problem* (1.1)–(1.3) *has a unique mild solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C(\tilde{Q})$, *where* $\tilde{Q}=\overline{Q}\setminus\{(t,a_1t):t\in[0,\infty)\}$.

The *proof* follows from Theorem 4.2.

The smoothness of a mild solution to problem (1.1)–(1.3) can also be increased if the matching condition (2.10) is true.

***Theorem* 6.2.** *Let the conditions be satisfied*

$$f\in C(\overline{Q}\times\mathbb{R}),\ \ \varphi\in C([0,\infty)),$$

$$\psi\in L^1_{\text{loc}}([0,\infty)),\ \ \mu\in C([0,\infty)),$$

and let the function f satisfy the Lipschitz condition with a function $L\in L^1_{\text{loc}}(\overline{Q})$ *with respect to the third variable, i. e.,* $|f(t,x,z)-f(t,x,w)|\le L(t,x)|z-w|$. *The first mixed problem* (1.1)–(1.3) *has a unique mild solution* $u:\overline{Q}\mapsto\mathbb{R}$ *in the class* $C(\overline{Q})$ *if and only if condition* (2.10) *is satisfied.*

Proof. In view of (4.1) and (4.2), we derive

$$u(t,x=at-0)-u(t,x=at+0)=\mu(0)-\varphi(0).$$

Since the constructed mild solution belongs to the class $C(\tilde{Q})$, it also belongs to the class $C(\overline{Q})$ if and only if $\mu(0)-\varphi(0)=0$. It proves the theorem.

The following theorem further improves the smoothness of the mild solution.

***Theorem* 6.3.** *Let the conditions be satisfied*

$$f\in C(\overline{Q}\times\mathbb{R}),\ \ \varphi\in C^1([0,\infty)),$$

$$\psi\in C([0,\infty)),\ \ \mu\in C^1([0,\infty)),$$

and let the function f satisfy the Lipschitz condition *with a function* $L \in L^1_{\text{loc}}(\bar{Q})$ *with respect to the third variable, i. e.,* $|f(t,x,z) - f(t,x,w)| \le L(t,x)|z - w|$. *The first mixed problem* (1.1)–(1.3) *has a unique mild solution* $u : \bar{Q} \mapsto \mathbb{R}$ *in the class* $C^1(\bar{Q})$ *if and only if conditions* (2.10) *and* (2.11) *are satisfied.*

Proof. In view of (4.1) and (4.2), we derive that $u \in C^1(\tilde{Q})$ under the smoothness conditions $f \in C(\bar{Q} \times \mathbb{R})$, $\varphi \in C^1([0,\infty))$, $\psi \in C([0,\infty))$, and $\mu \in C^1([0,\infty))$. Thus, this function u belongs to the class $C^1(\bar{Q})$ if and only if the following conditions are met:

$$\partial_t^p \partial_x^k u(t, x = at - 0) = \partial_t^p \partial_x^k u(t, x = at + 0), \quad (6.1)$$
$$p, k = 0,1, \quad 0 \le p + k \le 1.$$

Substituting (4.1) and (4.2) into conditions (6.1), we obtain equalities (2.10) and (2.11). The proof of the theorem is complete. □

Conclusions

In this paper, we derive the necessary and sufficient conditions for the existence of a unique classical solution to the first mixed problem in a quarter-plane for a semilinear wave equation with a mixed derivative. When the initial and boundary data are not sufficiently smooth, we construct a mild solution to the initial problem and prove its uniqueness.

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