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ДОКАЗАТЕЛЬСТВО ГИПОТЕЗЫ ОБ *F*-ИРРЕГУЛЯРНЫХ ГРАФАХ ДЛЯ ПРОСТОЙ ЦЕПИ P_n

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PROOF OF THE *F*-IRREGULAR GRAPH CONJECTURE FOR THE PATH P_n

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Аннотация. Пусть F и G – простые конечные неориентированные графы. Граф G называется F-иррегулярным, если любые две его различные вершины принадлежат различному числу подграфов из G, изоморфных графу F. В 1987 году Чартранд, Холберт, Оеллерман и Сварт выдвинули гипотезу о том, что для каждого связного графа F на трех и более вершинах существует нетривиальный F-иррегулярный граф. Мы подтверждаем эту гипотезу для каждой простой цепи P_n порядка $n \geq 3$. Кроме того, для любого целого числа $k \geq 6$ мы строим P_4-иррегулярный граф порядка k и показываем, что не существует нетривиального P_4-иррегулярного графа на пяти и менее вершинах.

Ключевые слова: *гипотеза об F-иррегулярных графах*, *простая цепь P_n*, *P_n-степень вершины*, *P_n-иррегулярный граф*, *(F, P_2)-иррегулярный граф.*

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Abstract. Let F and G be simple, finite, undirected graphs. A graph G is said to be F-irregular if any two of its distinct vertices belong to a different number of subgraphs of G that are isomorphic to F. In 1987, Chartrand, Holbert, Oellermann, and Swart conjectured that for every connected graph F with at least three vertices, there exists a nontrivial F-irregular graph. We confirm this conjecture for every path P_n of order $n \geq 3$. In addition, for each integer $k \geq 6$, we construct a P_4-irregular graph of order k and show that there does not exist a nontrivial P_4-irregular graph on five or fewer vertices.

Keywords: *F-irregular graph conjecture*, *path P_n*, *P_n-degree of a vertex*, *P_n-irregular graph*, *(F, P_2)-irregular graph*.

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Introduction

All the graphs considered in this paper are assumed to be simple, finite, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The order of G, denoted by $|G|$, is the number of its vertices, i. e., $|G| = |V(G)|$. A graph G is called nontrivial if $|G| \geq 2$. An edge incident to distinct vertices u and v is written as uv or (u, v). The degree of a vertex v in G is the number of edges incident to v. A graph is regular if all its vertices have the same degree. The neighborhood and closed neighborhood of a vertex v in G are defined as $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. A graph is connected if there exists a path between any pair of its vertices.

This paper investigates F-irregular graphs, a topic pioneered in 1987 by Chartrand, Holbert, Oellermann, and Swart [1] and based on a generalization of the classical notion of vertex degree.

***Definition* 0.1**. *For a graph F, the F-degree of a vertex v in a graph G, denoted by $Fdeg_G(v)$, is the number of subgraphs of G that are isomorphic to F and contain v. A graph G is called F-irregular if any two distinct vertices in G have different F-degrees*.

A central problem in this field is to characterize the connected graphs F for which a nontrivial F-irregular graph exists. Chartrand et al. [1] proved that stars and complete graphs of order at least 3 satisfy this condition and proposed the following conjecture.

***Conjecture* 0.1.** *For every connected graph F of order $|F| \geq 3$, there exists a nontrivial F-irregular graph*.

In 2024, Dovzhenok, Filuta, and Chuhai [2], for any biconnected graph F containing a vertex of degree 2, constructed an infinite series of F-irregular graphs and made a stronger statement than Conjecture 0.1.

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***Conjecture* 0.2.** *For each connected graph F with* $|F| \geq 3$, *there are infinitely many F-irregular graphs*.

Currently, Conjectures 0.1 and 0.2 remain open, and much of the research in this area focuses on the study of K_3-irregular graphs, where K_n is a complete graph on *n* vertices. Berikkyzy et al. [3] proved that the order of a K_3-irregular graph (triangle-distinct graph) can be any integer not less than 7, and discovered some structural properties of triangle-distinct graphs concerning their degrees and number of edges. Stevanović et al. [4] constructed regular K_3-irregular graphs, thereby providing an affirmative answer to the question of the existence of such graphs which was posed in 1988 by Chartrand, Erdős, and Oellermann [5].

In this article we explore the problem of *F*-irregular graphs in the case where *F* is a path.

***Definition* 0.2.** *Let* $n \geq 1$ *be an integer. A graph* P_n *with vertex set* $V(P_n) = \{v_1, v_2, ..., v_n\}$ *and edge set* $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$ *is called a path of order n. Vertices* v_1 *and* v_n *are the endpoints of* P_n. *The path* P_n *is denoted by* $v_1 v_2 ... v_n$ *or* $(v_1, v_2, ..., v_n)$, *where* $v_1 ... v_1 = v_1$ *by convention.*

It is evident that the degree and the P_2-degree of a vertex in a graph coincide. And since it is known that any nontrivial graph has at least two vertices with the same degree, it follows that a P_2-irregular graph does not exist. Figure 0.1 illustrates a P_3-irregular graph of order 6, with each vertex labeled by its P_3-degree. Moreover, Salehi [6] constructed a P_3-irregular graph of order *k* for every integer $k \geq 6$. This implies that Conjecture 0.2 holds for the path P_3.

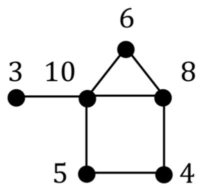

Figure 0.1 – P_3-irregular graph H_6 of order 6

In this paper, we confirm Conjecture 0.1 for each path P_n on $n \geq 3$ vertices (Section 2). We also construct a P_4-irregular graph of any order starting from 6, in particular, this confirms Conjecture 0.2 for a path P_4, and prove that a nontrivial P_4-irregular graph of order less than 6 does not exist (Section 1). Finally, in Section 3, we present the concept of (F, P_2)-irregular graphs.

The following basic statements will be useful in our work.

***Proposition* 0.1.** *If vertices u, v are symmetric in a graph G, i. e., there exists an automorphism f of G such that* $f(u) = v$, *then for any graph F, the vertices u, v have the same F-degrees in G.*

***Lemma* 0.1.** *Let F and G be graphs,* $u, v \in V(G)$, *with* $N_G(u) = N_G(v)$ *or* $N_G[u] = N_G[v]$. *Then* $Fdeg_G(u) = Fdeg_G(v)$.

Proof. From the condition of Lemma 0.1, it follows that there exists an automorphism *f* of *G* such that $f(u) = v$, $f(v) = u$, and $f(w) = w$ for each $w \in V(G) \setminus \{u, v\}$. Consequently, by Proposition 0.1, the vertices *u*, *v* have the same *F*-degrees in *G*. □

1 On the order of P_4-irregular graphs

We begin our investigation of the existence of P_n-irregular graphs by examining the case when $n = 4$, with our primary interest being the order of a P_4-irregular graph.

First, we will construct a P_4-irregular graph of any odd order, starting from 9.

***Definition* 1.1.** *Let* $l \geq 4$ *be an integer. We define a graph* H_{2l+1} *of order* $2l+1$ (*see Figure* 1.1) *as follows*:

$$V(H_{2l+1}) = V_1 \cup V_2 \cup \{2l+1\},$$

$$V_1 = \{1, 2, ..., l\},$$

$$V_2 = \{l+1, l+2, ..., 2l\},$$

$$E(H_{2l+1}) = \{(i, j) \mid i, j \in V_1, i < j\} \cup$$
$$\cup \{(i, j) \mid i \in V_2, j \in V_1, i - j \leq l\} \cup$$
$$\cup \{(1, 2l+1)\}.$$

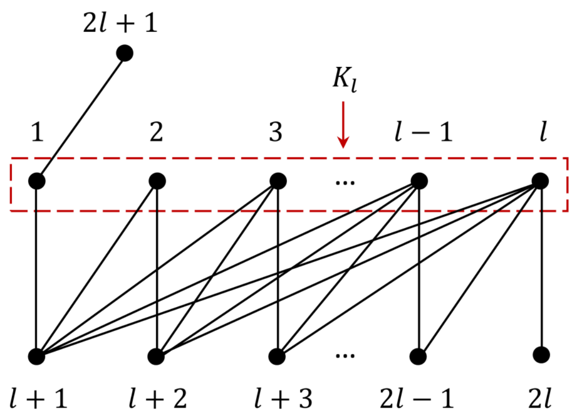

Figure 1.1 – Graph H_{2l+1}

***Theorem* 1.1.** *For every integer* $l \geq 4$, *the graph* H_{2l+1} *is* P_4*-irregular.*

Proof. Fix an integer $l \geq 4$ and consider the graph H_{2l+1}. We denote by *P* the set of all subgraphs of H_{2l+1} that are isomorphic to P_4, and let

$$h_i = P_4deg_{H_{2l+1}}(i) \quad \forall i \in V(H_{2l+1}).$$

We show that the P_4-degrees of vertices in H_{2l+1} satisfy the following inequalities:

$$h_{2l} > h_{2l+1}, \quad (1.1)$$

$$h_i > h_{i+1} \quad \forall i \in \{l+1, ..., 2l-1\}, \quad (1.2)$$

$$h_1 > h_{l+1}, \quad (1.3)$$

$$h_2 > h_1, \quad (1.4)$$

$$h_{i+1} > h_i \quad \forall i \in \{2, ..., l-1\}. \quad (1.5)$$

First, we calculate h_{2l}. There are exactly 3 disjoint sets of copies of P_4 in H_{2l+1} that contain vertex $2l$:

$$A_1 = \{(2l, l, u, v) \mid u \in V_2 \setminus \{2l\}, v \in \{u-l, ..., l-1\}\},$$

$$A_2 = \{(2l, l, u, v) \mid u \in V_1 \setminus \{l\}, v \in \{1, ..., l+u\} \setminus \{l, u\}\},$$

$$A_3 = \{(2l, l, 1, 2l+1)\}.$$

Therefore,

$$h_{2l} = |A_1| + |A_2| + |A_3| =$$

$$= \sum_{u=l+1}^{2l-1}(2l-u) + \sum_{u=1}^{l-1}(l+u-2) + 1 =$$

$$= \frac{l(l-1)}{2} + \frac{(3l-4)(l-1)}{2} + 1 =$$

$$= 2(l-1)^2 + 1. \quad (1.6)$$

Next, we find h_{2l+1}. We have exactly 2 disjoint sets of copies of P_4 in H_{2l+1} that contain vertex $2l+1$:

$$B_1 = \{(2l+1, 1, l+1, u) \mid u \in V_1 \setminus \{1\}\},$$

$$B_2 = \{(2l+1, 1, u, v) \mid u \in V_1 \setminus \{1\}, v \in \{2, ..., l+u\} \setminus \{u\}\}.$$

Therefore,

$$h_{2l+1} = |B_1| + |B_2| =$$

$$= l - 1 + \sum_{u=2}^{l}(l+u-2) =$$

$$= l - 1 + \frac{(3l-2)(l-1)}{2} = \frac{3l(l-1)}{2}. \quad (1.7)$$

From (1.6), (1.7), and the inequality $l \geq 4$, we have (1.1):

$$h_{2l} > h_{2l+1} \Leftrightarrow (l-1)(l-4) + 2 > 0.$$

Next, we prove (1.2). Fix $i \in \{l+1, ..., 2l-1\}$. Let H (see Figure 1.2) be the graph formed from H_{2l+1} by removing the edge $(i, i-l)$.

Note that vertices i and $i+1$ have the same neighborhoods in H:

$$N_H(i) = N_H(i+1) = \{i+1-l, ..., l\}.$$

Therefore, by Lemma 0.1,

$$P_4deg_H(i) = P_4deg_H(i+1).$$

For the graph H_{2l+1}, this means that vertices i and $i+1$ belong to the same number of subgraphs of H_{2l+1} that are isomorphic to P_4 and do not contain the edge $(i, i-l)$. Thus,

$$h_i - h_{i+1} = |X|, \; i \in \{l+1, ..., 2l-1\},$$

$$X = \{F \in P \mid (i, i-l) \in E(F), i+1 \notin V(F)\}. \quad (1.8)$$

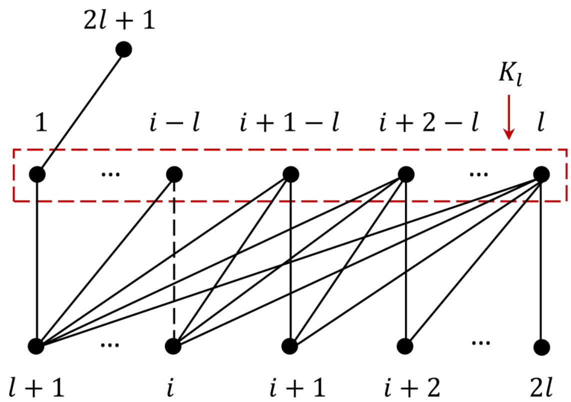

Figure 1.2 – Graph $H = H_{2l+1} \setminus \{(i, i-l)\}$

And since the path $(i, i-l, x, y) \in X$, where $x, y \in V_1 \setminus \{i-l\}, x < y$, then $|X| > 0$, and together with (1.8) this guarantees the truth of (1.2).

Next, to prove (1.3), consider the graph G which is formed from H_{2l+1} by removing the edge $(1, 2l+1)$. Note that the closed neighborhoods of vertices 1 and $l+1$ coincide in graph G:

$$N_G[1] = N_G[l+1] = \{1, ..., l+1\}.$$

Therefore, by Lemma 0.1,

$$P_4deg_G(1) = P_4deg_G(l+1),$$

and hence,

$$h_1 - h_{l+1} = |Y|,$$

$$Y = \{F \in P \mid (1, 2l+1) \in E(F), l+1 \notin V(F)\}. \quad (1.9)$$

For (1.3) to be true, taking into account (1.9), it remains to note that $|Y| > 0$, since the path $(2l+1, 1, 2, 3) \in Y$.

We now prove (1.4). Let W (see Figure 1.3) be the graph obtained from H_{2l+1} by removing the two edges $(1, 2l+1)$ and $(2, l+2)$. Note that the closed neighborhoods of vertices 1 and 2 in the graph W are equal:

$$N_W[1] = N_W[2] = \{1, ..., l+1\}.$$

Therefore, by Lemma 0.1,

$$P_4deg_W(1) = P_4deg_W(2),$$

and hence,

$$h_2 - h_1 = |M| - |K|,$$

$$K = \{F \in P \mid (1, 2l+1) \in E(F), 2 \notin V(F)\},$$

$$M = \{F \in P \mid (2, l+2) \in E(F), 1 \notin V(F)\}. \quad (1.10)$$

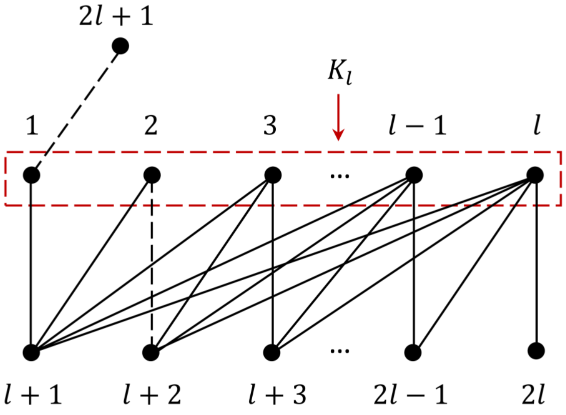

Figure 1.3 – Graph $W = H_{2l+1} \setminus \{(1, 2l+1), (2, l+2)\}$

The set M can be represented as the union of five disjoint sets:

$$M_1 = \{(l+2, 2, l+1, v) \mid v \in \{3, ..., l\}\},$$

$$M_2 = \{(l+1, 2, l+2, v) \mid v \in \{3, ..., l\}\},$$

$$M_3 = \{(2, l+2, u, v) \mid u \in \{3, ..., l\}, \; v \in \{3, ..., l+u\} \setminus \{u, l+2\}\},$$

$$M_4 = \{(l+2, 2, u, v) \mid u \in \{3, ..., l\}, \; v \in \{3, ..., l+u\} \setminus \{u, l+2\}\},$$

$$M_5 = \{(u, 2, l+2, v) \mid u, v \in \{3, ..., l\}, u \neq v\}.$$

Therefore,

$$|M| = |M_1| + |M_2| + |M_3| + |M_4| + |M_5| = = 2(l-2) + 2\sum_{u=3}^{l}(l+u-4) + (l-2)(l-3) = = (l-2)(4l-6). \quad (1.11)$$

Then, taking into account (1.7), (1.10), (1.11), and the inequality $l \geq 4$, we have

$$h_2 - h_1 = |M| - |K| \geq |M| - h_{2l+1} = = (l-2)(4l-6) - \frac{3l(l-1)}{2} = = \frac{1}{2}(5l^2 - 25l + 24) > 0,$$

from which (1.4) follows.

Finally, we prove (1.5). Fix $i \in \{2, ..., l-1\}$ and similarly to proof (1.2), we obtain

$$h_{i+1} - h_i = |Z|, \; i \in \{2, ..., l-1\},$$

$$Z = \{F \in P \mid (i+1, i+l+1) \in E(F), i \notin V(F)\}. \quad (1.12)$$

Note that the path $(i+l+1, i+1, z, w) \in Z$, where $z, w \in V_1 \setminus \{i, i+1\}, z < w$. Therefore, $|Z| > 0$ and, consequently, according to (1.12), inequality (1.5) is true.

From (1.1)–(1.5) it follows that the P_4-degrees of the vertices in H_{2l+1} are pairwise distinct. Therefore, the graph H_{2l+1} is P_4-irregular. □

Now we will formulate a criterion for the existence of a nontrivial P_4-irregular graph of order *k*.

***Theorem* 1.2.** *There exists a nontrivial P_4-irregular graph of order k if and only if k is an integer and $k \geq 6$.*

Proof. Sufficiency. Figure 1.4 shows the P_4-irregular graphs H_6, H_7, of order 6, 7, respectively, and each vertex of these graphs is labeled with its P_4-degree. An example of a P_4-irregular graph of odd order $k \geq 9$, according to Theorem 1.1, is the graph H_k. A P_4-irregular graph of even order $k \geq 8$ can be constructed by adding an isolated vertex to the graph H_{k-1}, whose P_4-degrees of vertices, based on Figure 1.4 and the proof of Theorem 1.1, are distinct and positive.

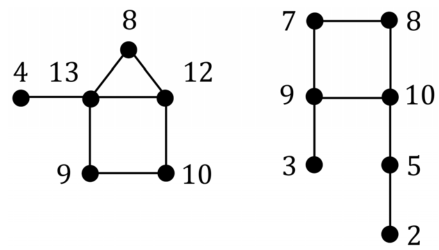

Figure 1.4 – P_4-irregular graphs H_6, H_7 of order 6, 7

Necessity. Consider any nontrivial P_4-irregular graph of order *k*. By definition, *k* is an integer. Moreover, $k > 4$, since in any graph of order 2, 3, or 4, the P_4-degrees of all vertices are equal. Next, referring to the graph diagrams [7, Appendix 1, P. 216–217], it is straightforward to verify that each of the 34 existing graphs of order 5 has two symmetric vertices. Consequently, by Proposition 0.1, the P_4-degrees of such vertices are equal in every graph of order 5, which implies that no P_4-irregular graph of order 5 exists. From the foregoing, it follows that $k \geq 6$. This completes the proof. □

2 Main result

Our goal in this section is to confirm Conjecture 0.1 for any path P_n with $n \geq 3$ vertices.

***Theorem* 2.1.** *For every integer $n \geq 3$, there exists a nontrivial P_n-irregular graph.*

Proof. Since the graph H_6 (Figures 0.1, 1.4) is P_3-irregular and P_4-irregular, Theorem 2.1 is true for $n = 3$ and $n = 4$.

Let $n \geq 5$ be an integer. Consider a graph S_{3n-4} (Figure 2.1) with a set of vertices

$$V(S_{3n-4}) = V_1 \cup V_2 \cup V_3,$$

$$V_1 = \{a_1, a_2, ..., a_{n-1}\},$$

$$V_2 = \{b_1, b_2, ..., b_n\},$$

$$V_3 = \{c_1, c_2, ..., c_{n-3}\},$$

and a set of ages

$$E(S_{3n-4}) = \{a_i a_{i+1} \mid 1 \le i \le n-2\} \cup$$
$$\cup \{b_i b_{i+1} \mid 1 \le i \le n-1\} \cup$$
$$\cup \{c_i c_{i+1} \mid 1 \le i \le n-4\} \cup \{a_{n-1}b_n, a_{n-1}c_{n-3}, b_n c_{n-3}\}.$$

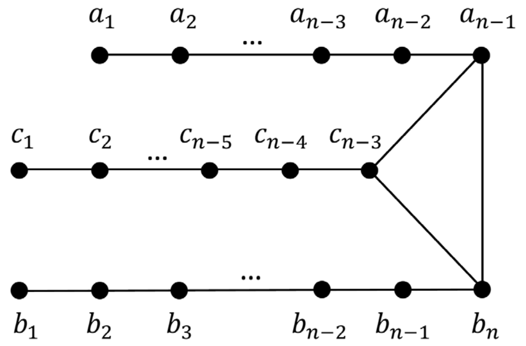

Figure 2.1 – Graph S_{3n-4}

Let us prove that S_{3n-4} is a P_n-irregular graph.

We define an n-path as any subgraph of S_{3n-4} that is isomorphic to P_n.

First, we find the P_n-degrees of vertices from the set V_1.

The vertex a_1 belongs to exactly two n-paths: $a_1 \ldots a_{n-1}b_n$, $a_1 \ldots a_{n-1}c_{n-3}$. Therefore,

$$P_n deg_{S_{3n-4}}(a_1) = 2. \quad (2.1)$$

Further, let $i \in \{2, ..., n-3\}$. The graph S_{3n-4} contains exactly 4 n-paths with endpoint a_i:

$$a_i \ldots a_{n-1}c_{n-3} \ldots c_{n-i-2},$$
$$a_i \ldots a_{n-1}c_{n-3}b_n \ldots b_{n-i+2},$$
$$a_i \ldots a_{n-1}b_n \ldots b_{n-i+1},$$
$$a_i \ldots a_{n-1}b_n c_{n-3} \ldots c_{n-i-1}. \quad (2.2)$$

Note that any n-path containing a vertex a_i either contains a vertex a_{i-1}, or, alternatively, has a_i as an endpoint. From this, taking into account (2.1), (2.2), we obtain

$$P_n deg_{S_{3n-4}}(a_i) = P_n deg_{S_{3n-4}}(a_{i-1}) + 4$$
$$\forall i \in \{2, \ldots, n-3\}, \ P_n deg_{S_{3n-4}}(a_1) = 2.$$

Thus, the P_n-degrees of the vertices $a_1, \ldots, a_{n-3}$ in S_{3n-4} form an arithmetic progression with a difference of 4 and a first term of 2. Therefore,

$$P_n deg_{S_{3n-4}}(a_i) = 4i - 2 \ \ \forall i \in \{1, \ldots, n-3\}. \quad (2.3)$$

Consider the vertex a_{n-2}. The graph S_{3n-4} contains exactly 3 n-paths with endpoint a_{n-2}:

$$a_{n-2}a_{n-1}c_{n-3}b_n \ldots b_4,$$
$$a_{n-2}a_{n-1}b_n \ldots b_3,$$
$$a_{n-2}a_{n-1}b_n c_{n-3} \ldots c_1. \quad (2.4)$$

Similar to finding the P_n-degrees of vertices $a_2, ..., a_{n-3}$, taking into account (2.3), (2.4), we deduce that

$$P_n deg_{S_{3n-4}}(a_{n-2}) =$$
$$= P_n deg_{S_{3n-4}}(a_{n-3}) + 3 = 4n - 11. \quad (2.5)$$

Let's look at the vertex a_{n-1}. Given the structure of S_{3n-4}, the set of all n-paths containing a_{n-1} can be represented as the union of two disjoint subsets G_1 and G_2, where G_1 is the set of all n-paths containing a_{n-2}, and G_2 is the set of all n-paths containing a_{n-1} and not having vertices in $V_1 \setminus \{a_{n-1}\}$. From (2.5), it follows that

$$|G_1| = P_n deg_{S_{3n-4}}(a_{n-2}) = 4n - 11. \quad (2.6)$$

Let us calculate $|G_2|$. Clearly, for every n-path from G_2, one of its endpoints, but not both, belongs to the set $V_2 \setminus \{b_1, b_n\}$.

Next, for each $i \in \{2, \ldots, n-1\}$, we list all n-paths with endpoint b_i that contain a_{n-1} and do not contain vertices from $V_1 \setminus \{a_{n-1}\}$:

$$b_2 : \ b_2 \ldots b_n a_{n-1},$$
$$b_3 : \ b_3 \ldots b_n a_{n-1} c_{n-3}, \ b_3 \ldots b_n c_{n-3} a_{n-1},$$
$$b_i : \ b_i \ldots b_n a_{n-1} c_{n-3} \ldots c_{n-i} \ \ \forall i \in \{4, \ldots, n-1\}. \quad (2.7)$$

From (2.7), it follows that

$$|G_2| = n - 1. \quad (2.8)$$

Therefore, using (2.6) and (2.8), we obtain

$$P_n deg_{S_{3n-4}}(a_{n-1}) = |G_1| + |G_2| = 5n - 12. \quad (2.9)$$

Let us find P_n-degrees of vertices from the set V_2. Since the vertex b_1 belongs to only one n-path $(b_1 \ldots b_n)$, then

$$P_n deg_{S_{3n-4}}(b_1) = 1. \quad (2.10)$$

Let H_1 be the graph obtained from S_{3n-4} by removing the vertex b_1. It is easy to see that in H_1 the vertices $b_2, \ldots, b_n$ are symmetric to the vertices $a_1, \ldots, a_{n-1}$, respectively.

Therefore, by Proposition 0.1, we have

$$P_n deg_{H_1}(b_i) = P_n deg_{H_1}(a_{i-1}) \ \ \forall i \in \{2, \ldots, n\}. \quad (2.11)$$

And since when deleting b_1 from S_{3n-4}, exactly one n-path $b_1 \ldots b_n$ is "lost", then

$$P_n deg_{H_1}(a_i) = P_n deg_{S_{3n-4}}(a_i) \ \ \forall i \in \{1, \ldots, n-1\},$$
$$P_n deg_{H_1}(b_i) = P_n deg_{S_{3n-4}}(b_i) - 1 \ \ \forall i \in \{2, \ldots, n\}. \quad (2.12)$$

Taking into account (2.3), (2.5), (2.9), (2.11), and (2.12), we have

$$P_n deg_{S_{3n-4}}(b_i) = P_n deg_{S_{3n-4}}(a_{i-1}) + 1 = $$
$$= 4i - 5 \quad \forall i \in \{2, \ldots, n-2\},$$
$$P_n deg_{S_{3n-4}}(b_{n-1}) =$$
$$= P_n deg_{S_{3n-4}}(a_{n-2}) + 1 = 4n - 10,$$
$$P_n deg_{S_{3n-4}}(b_n) =$$
$$= P_n deg_{S_{3n-4}}(a_{n-1}) + 1 = 5n - 11. \qquad (2.13)$$

Finally, we find the P_n-degrees of the vertices from the set V_3. Let H_2 be the graph obtained from S_{3n-4} by removing vertices a_1 and a_2. In H_2, the vertices $c_1, \ldots, c_{n-3}$ are symmetric to the vertices $a_3, \ldots, a_{n-1}$, respectively. Therefore, by Proposition 0.1, we have

$$P_n deg_{H_2}(c_i) = P_n deg_{H_2}(a_{i+2})$$
$$\forall i \in \{1, \ldots, n-3\}. \qquad (2.14)$$

Note that when deleting vertices a_1, a_2 from S_{3n-4}, exactly 6 n-paths were "lost":

$$a_1 \ldots a_{n-1} b_n, \; a_1 \ldots a_{n-1} c_{n-3},$$
$$a_2 \ldots a_{n-1} b_n b_{n-1}, \; a_2 \ldots a_{n-1} b_n c_{n-3},$$
$$a_2 \ldots a_{n-1} c_{n-3} b_n, \; a_2 \ldots a_{n-1} c_{n-3} c_{n-4}. \qquad (2.15)$$

From (2.15), it follows that

$$P_n deg_{H_2}(a_i) = P_n deg_{S_{3n-4}}(a_i) - 6$$
$$\forall i \in \{3, \ldots, n-1\},$$
$$P_n deg_{H_2}(c_i) = P_n deg_{S_{3n-4}}(c_i)$$
$$\forall i \in \{1, \ldots, n-5\},$$
$$P_n deg_{H_2}(c_{n-4}) = P_n deg_{S_{3n-4}}(c_{n-4}) - 1,$$
$$P_n deg_{H_2}(c_{n-3}) = P_n deg_{S_{3n-4}}(c_{n-3}) - 4. \qquad (2.16)$$

Based on (2.3), (2.5), (2.9), (2.14), and (2.16), we come to the conclusion that

$$P_n deg_{S_{3n-4}}(c_i) = P_n deg_{S_{3n-4}}(a_{i+2}) - 6 = 4i$$
$$\forall i \in \{1, \ldots, n-5\},$$
$$P_n deg_{S_{3n-4}}(c_{n-4}) = P_n deg_{S_{3n-4}}(a_{n-2}) - 5 = 4n - 16,$$
$$P_n deg_{S_{3n-4}}(c_{n-3}) =$$
$$= P_n deg_{S_{3n-4}}(a_{n-1}) - 2 = 5n - 14. \qquad (2.17)$$

To complete the proof of Theorem 2.1, it remains to note that for any integer $n \geq 5$, according to (2.3), (2.5), (2.9), (2.10), (2.13), and (2.17), the P_n-degrees of all vertices in S_{3n-4} are pairwise distinct. Consequently, S_{3n-4} is a P_n-irregular graph for every integer $n \geq 5$. □

Conclusion

In this paper, for each integer $k \geq 6$, we constructed a P_4-irregular graph of order k and proved that for every path P_n on $n \geq 3$ vertices, there exists a nontrivial P_n-irregular graph.

Ali, Chartrand, and Zhang [8, p. 31] suggest several other research directions based on variations of the *F*-degree of a vertex in a graph. In relation to paths, we highlight for further study the problem of the existence of $[P_n]$*-irregular graphs*. In such graphs, any two distinct vertices belong to a different number of *induced* subgraphs isomorphic to the path P_n.

We also present a new concept of irregularity in graphs, shifting the focus from the vertices of a graph to its edges. For graphs *F* and *G*, we say that *G* is (F, P_2)*-irregular* if any two distinct edges of the graph *G* belong to different numbers of subgraphs of *G* that are isomorphic to *F*. Specifically, regarding paths, we pose the following problem: *for which integers* $n \geq 4$, *does a* (P_n, P_2) *-irregular graph with two or more edges exist*?

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