Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. We use $\mathfrak{U}$ to denote the class of all supersoluble groups; $G^N$ denotes the intersection of all normal subgroups $N$ of $G$ with $G / N \in \mathfrak{U}$. The symbol $\pi(G)$ denotes the set of prime divisors of the order of $G$.

A subgroup $H$ of $G$ is called a 2-maximal (second maximal) subgroup of $G$ whenever $H$ is a maximal subgroup of some maximal subgroup $M$ of $G$. Similarly we can define 3-maximal subgroups, and so on. If $H$ is $n$-maximal in $G$ but not $n$-maximal in any proper subgroup of $G$, then $H$ is said to be a strictly $n$-maximal subgroup of $G$.

One of the interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its $n$-maximal subgroups. The earliest publications in this direction are the articles of L. Rédei [1] and B. Huppert [2]. L. Rédei described the nonsoluble groups with abelian two maximal subgroups. B. Huppert established the supersolubility of $G$ whose all second maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of $G$ are normal in $G$, then the commutator subgroup $G'$ of $G$ is nilpotent and the chief rank of $G$ is at most 2. These results were developed by many authors. In particular, L.Ja. Poljakov [3] proved that $G$ is supersoluble if every 2-maximal subgroup of $G$ is permutable with every maximal subgroup of $G$. He also established the solubility of $G$ in the case when every maximal subgroup of $G$ permutes with every 3-maximal subgroups of $G$.

Some later, R.K. Agrawal [4] proved that $G$ is supersoluble if any 2-maximal subgroup of $G$ is permutable with every Sylow subgroup of $G$. In [5], Z. Janko described the groups whose 4-maximal subgroups are normal. A description of nonsoluble groups with all 2-maximal subgroups nilpotent was obtained by M. Suzuki [6] and Z. Janko [7]. In [8], T.M. Gagen and Z. Janko gave a description of simple groups whose 3-maximal subgroups are nilpotent. V.A. Belonogov [9] studied those groups in which every 2-maximal subgroup is nilpotent. Continuing this, V.N. Semenchuk [10] obtained a description of soluble groups whose all 2-maximal subgroups are supersoluble. A. Mann [11] studied the structure of the groups whose $n$-maximal subgroups are subnormal. He proved that if all $n$-maximal subgroups of a soluble group $G$ are subnormal and $|\pi(G)| \geq n + 1$, then $G$ is nilpotent; but if $|\pi(G)| \geq n - 1$, then $G$ is $\varphi$-dispersive for some ordering $\varphi$ of the set of all primes. Finally, in the case $|\pi(G)| = n$, Mann described $G$ completely.

A.E. Spencer [12] studied groups in which every $n$-maximal chain contains subnormal subgroup. In particular, Spencer proved that $G$ is a Schmidt group with abelian Sylow subgroups if every 2-maximal chain of $G$ contains subnormal subgroup. In [13], M. Asaad studied groups whose strictly...
Among the recent results on $n$-maximal subgroups we can mention the paper of Y.V. Guo and K.P. Shum [15]. In this paper the authors proved that $G$ is soluble if all its $2$-maximal subgroups enjoy the cover-avoidance property. W. Guo, K.P. Shum, A.N. Skiba and L. Baojun [16, 17, 18] gave new characterizations of supersoluble groups in terms of $2$-maximal subgroups. Li Shirong [19] obtained a classification of non-nilpotent groups whose all $2$-maximal subgroups are $TI$-subgroups. In the paper [20], W. Guo, H.V. Legchevka and A.N. Skiba described the groups whose every $3$-maximal subgroup permutes with all maximal subgroups. In [21], W. Guo, Yu.V. Lutsenko and A.N. Skiba gave a description of non-nilpotent groups in which every $3$-maximal subgroup permutes with all $2$-maximal subgroups. Moreover, in [22], W. Guo, D.P. Andreeva and A.N. Skiba obtained the description of the groups whose all $3$-maximal subgroups are $S$-quasinormal. Subsequently, this result was strengthened by Yu.V. Lutsenko and A.N. Skiba in [23] to provide a description of the groups whose all $3$-maximal subgroups are subnormal. Developing some of the above-mentioned results, D.P. Andreeva and A.N. Skiba [24] obtained a description of the groups in which every $3$-maximal chain contains a proper $S$-quasinormal subgroup. Moreover, in [25], W. Guo, D.P. Andreeva and A.N. Skiba obtained the description of the groups in which every $3$-maximal chain contains a proper subnormal subgroup. In [26], A. Ballester-Bolívarches, L.M. Ezquero and A.N. Skiba obtained a full classification of the groups in which the second maximal subgroups of the Sylow subgroups cover or avoid the chief factors of some of its chief series. In [27], V.N. Kniazhina and V.S. Monakhov studied those groups $G$ in which every $n$-maximal subgroup permutes with each Schmidt subgroup. In particular, it was proved that if $n = 1, 2, 3$, then $G$ is metanilpotent; but if $n \geq 4$ and $G$ is soluble, then the nilpotent length of $G$ is at most $n - 1$.

Another interesting results on $n$-maximal subgroups were obtained by V.A. Kovaleva and A.N. Skiba in [28], [29] and V.S. Monakhov and V.N. Kniazhina in [30]. In [28], the authors described the groups whose all $n$-maximal subgroups are $\mathcal{U}$-subnormal. In [29] a description of the groups with all $n$-maximal subgroups $\mathcal{U}$-subnormal for some saturated formation $\mathfrak{S}$ was obtained. In [30], the groups with all $2$-maximal subgroups $\mathcal{P}$-subnormal were studied.

Recall that a subgroup $H$ of $G$ is said to be:

(i) $\mathcal{U}$-subnormal in $G$ if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that $H_i/\langle H_{i-1}\rangle \in \mathcal{U}$, for $i = 1, \ldots, n$;

(ii) $\mathcal{U}$-subnormal (in the sense of Kegel [31]) or $K\mathcal{V}$-subnormal [32, p. 236] in $G$ if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_t = G$$

such that either $H_{i+1}$ is normal in $H_i$ or $H_i/\langle H_{i-1}\rangle \in \mathcal{U}$ for all $i = 1, \ldots, t$. It is evident that every subnormal subgroup is $K\mathcal{V}$-subnormal. The inverse, in general, is not true. For example, in the group $S_5$ a subgroup of order $2$ is $K\mathcal{V}$-subnormal and at the same time it is not subnormal.

The elementary observation and the results in [23], [25] make natural the following question:

I. What is the structure of $G$ under the condition that every $2$-maximal subgroup of $G$ is $K\mathcal{V}$-subnormal?

II. What is the structure of $G$ under the condition that every $3$-maximal subgroup of $G$ is $K\mathcal{V}$-subnormal?

In this paper we given the solutions of these two questions.

1 Preliminary results

The solutions of Question I and Question II are based on the following results.

**Lemma 1.1.** Let $H$ and $K$ be subgroups of $G$ such that $H$ is $K\mathcal{V}$-subnormal in $G$.

(1) $H \cap K$ is $K\mathcal{V}$-subnormal in $K$ [32, Lemma 6.1.7 (2)].

(2) If $N$ is a normal subgroup in $G$, then $HN/N$ is $K\mathcal{V}$-subnormal in $G/N$ [32, Lemma 6.1.6 (3)].

(3) If $K$ is $K\mathcal{V}$-subnormal in $H$, then $K$ is $K\mathcal{V}$-subnormal in $G$ [32, Lemma 6.1.6 (1)].

(4) If $G^2 \leq K$, then $K$ is $K\mathcal{V}$-subnormal in $G$ [32, Lemma 6.1.7 (1)].

The next lemma is evident.

**Lemma 1.2.** If $G$ is supersolvable, then every subgroup of $G$ is $K\mathcal{V}$-subnormal in $G$.

**Lemma 1.3.** If every $n$-maximal subgroup of $G$ is $K\mathcal{V}$-subnormal in $G$, then every $(n-1)$-maximal subgroup of $G$ is supersolvable and every $(n+1)$-maximal subgroup of $G$ is $K\mathcal{V}$-subnormal in $G$.

**Proof.** We first show that every $(n-1)$-maximal subgroup of $G$ is supersolvable. Let $H$ be an $(n-1)$-maximal subgroup of $G$ and $K$ any maximal subgroup of $H$. Then $K$ is an $n$-maximal subgroup of $G$ and so, by hypothesis, $K$ is $K\mathcal{V}$-subnormal in $G$. Hence $K$ is $K\mathcal{V}$-subnormal in $H$ by Lemma 1.1 (1). Therefore either $K$ is normal in $H$ or $H/K$ is $K\mathcal{V}$-subnormal in $G$. If $K$ is normal in $H$, then $|H:K|$ is a prime in view of maximality of $K$ in $H$. Let $H/K$ be $K\mathcal{V}$-subnormal. Then we also get that

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Finite groups with all n-maximal \((n = 2, 3)\) subgroups \(K - \Omega\)-subnormal

If every 2-maximal subgroup of \(G\) is \(\Omega\)-subnormal in \(G\) and only if \(G\) is either supersoluble or an SDH-group.

If every 2-maximal subgroup of \(G\) is \(K - \Omega\)-subnormal, then every maximal subgroup of \(G\) is supersoluble by Lemma 1.3. Therefore in this case \(G\) is either supersoluble or a minimal nonsupersoluble group, hence \(G\) is soluble by [2]. Thus we get the following

Theorem A. Every 2-maximal subgroup of \(G\) is \(K - \Omega\)-subnormal in \(G\) and only if \(G\) is either supersoluble or an SDH-group.

The solution of Question II is more complete. Note that since each subgroup of every supersoluble group is \(K - \Omega\)-subnormal, we need, in fact, only consider the case when \(G\) is not supersoluble. But in this case, in view of [28] or [29], \(|\pi(G)| \leq 4\).

The following theorems are proved.

Theorem B. Let \(G\) be a nonsupersoluble group with \(|\pi(G)| = 2\). Let \(p, q\) be distinct prime divisors of \(|G|\) and \(pG, qG\) be Sylow \(p\)-subgroup and \(q\)-subgroup of \(G\) respectively. Every 3-maximal subgroup of \(G\) is \(K - \Omega\)-subnormal in \(G\) and only if \(G\) is a solvable group of one of the following types:

I. \(G\) is a minimal nonsupersoluble group and either \(|\Phi(G^p)|\) is a prime or \(|\Phi(G^q)| = 1\).

II. \(G = pG \times qG\), where \(pG\) is the unique minimal normal subgroup of \(G\) and every 2-maximal subgroup of \(G\) is an Abelian group of exponent dividing \(p - 1\). Moreover, every maximal subgroup of \(G\) containing \(pG\) is either supersoluble or an SDH-group and at least one of the maximal subgroup of \(G\) is not supersoluble.

III. \(G = (pG \times QG) \times QG\), where \(pG \times QG\) and \(QG\) are minimal normal subgroups of \(G\), \(|QG| = q\), \(pG \times QG\) is an SDH-group and every maximal subgroup of \(G\) containing \(pG \times QG\) is supersoluble. Moreover, if \(p < q\), then every 2-maximal subgroup of \(G\) is nilpotent.

IV. \(G = pG \times qG\), where \(pG\) is a minimal normal subgroup of \(G\), \(QG(G) \neq 1\), \(\Phi(G) \neq 1\), every maximal subgroup of \(G\) containing \(pG\) is either supersoluble or an SDH-group and \(G/\Phi(G)\) is a group one of types II or III.

V. \(G = (P_1 \times P_2) \times G_q\), where \(pG = P_1 \times P_2\), \(P_1, P_2\) are minimal normal subgroups of \(G\), every maximal subgroup of \(G\) containing \(pG\) is supersoluble, \(P_1 \times G_q\) is an SDH-group and \(P_2 \times G_q\) is either an SDH-group or a supersoluble group with \(|P_2| = p|\).

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VI. $G = G_p \times G_q$, $\Phi(G_p)$ is a minimal normal subgroup of $G$, every maximal subgroup of $G$ containing $G_p$ is supersoluble and $\Phi(G_p) \times G_q$ is an SDH-group.

VII. Every of the subgroups $G_p$ and $G_q$ is not normal in $G$ and the following hold:

(i) if $p < q$, then $G = P_1 \times (G_q \times P_2)$, where $G_p = P_1 \times P_2$, $P_1$ is a minimal normal subgroup of $G$, $|P_2| = p$, $G_q = \langle a \rangle$ is a cyclic group and $\langle a^n \rangle$ is normal in $G$. Moreover, $G$ has precisely three classes of maximal subgroups whose representatives are $P_1 \times G_q$, $G_q \times P_2$, $\langle a^n \rangle \times G_q$, where $P_1 \times G_q$ is an SDH-group

(ii) if $p > q$, then $G = P_2 \times (G_q \times P_1)$, where $G_p = P_1 \times P_2$, $P_2$ is a normal subgroup of $G$, $P_2 = \langle b \rangle$ is a cyclic group and $1 \neq P_1 \cap P_2 = \langle b^n \rangle$. Moreover, $G$ has precisely three classes of maximal subgroups whose representatives are $P_1 \times G_q$, $G_q \times P_2$, $G_p$, where $|G : G_q \times P_2| = p$, $P_1 \times G_q$ is a supersoluble group and $G_q \times P_2$ is an SDH-group.

Theorem C. Let $G$ be a nonsupersoluble group with $|\pi(G)| = 3$. Let $p, q, r$ be distinct prime divisors of $|G|$ and $G_p$, $G_q$, $G_r$ be Sylow $p$-subgroup, $q$-subgroup and $r$-subgroup of $G$ respectively. Every 3-maximal subgroup of $G$ is $K - \Omega$-subnormal in $G$ if and only if $G$ is a soluble group with all 3-maximal subgroups subnormal [23].

Theorem D. Let $G$ be a nonsupersoluble group with $|\pi(G)| = 4$. Let $p, q, r, t$ be distinct prime divisors of $|G|$ ($p > q > r > t$) and $G_p$, $G_q$, $G_r$, $G_t$ be Sylow $p$-subgroup, $q$-subgroup, $r$-subgroup and $t$-subgroup of $G$ respectively. Every 3-maximal subgroup of $G$ is $K - \Omega$-subnormal in $G$ if and only if $G = G_p \times (G_q \times (G_r \times G_t))$ is a soluble group such that $G$ has precisely three classes of maximal subgroups whose representatives are $G_pG_qG_r$, $G_pG_q\Phi(G_t)$, $G_pG_r\Phi(G_t)$ and $G_r\Phi(G_pG_t)$, and every nonsupersoluble maximal subgroup of $G$ is an SDH-group, $G_p$ and $G_t$ are cyclic groups and following hold:

(1) if $G_pG_qG_r$ is an SDH-group, then $G_p = G_q \times G_r$ and $G_q$ is a minimal normal subgroup of $G$, the subgroups $G_pG_q\Phi(G_t)$ and $G_pG_r\Phi(G_t)$ are supersoluble and $G_r\Phi(G_pG_t)$ is either an SDH-group or a supersoluble group with $|G_p| = p$;

(2) if $G_pG_qG_t$ is a soluble group, then $G_q$ is cyclic.

The classes of groups which are described in Theorems B and C are pairwise disjoint. It is easy to construct examples to show that all classes of the groups in this theorems and in Theorems A and D are not empty. Note also that Theorems B, C and D show that the class of the groups with all 3-maximal subgroups $K - \Omega$-subnormal is essentially wider than the class of the groups with all 3-maximal subgroups subnormal [23].

REFERENCES


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