МАТЕМАТИКА

УДК 512.542 DOI: https://doi.org/10.54341/20778708_2025_4_65_72
EDN: KEQNCF

ОБ ОДНОМ КЛАССЕ ПОДРЕШЕТОК РЕШЕТКИ ПОДГРУПП КОНЕЧНОЙ ГРУППЫ

Н.С. Косенок, И.В. Близнец

Гомельский государственный университет имени Франциска Скорины

ON A CLASS OF SUBLATTICES OF THE SUBGROUP LATTICE OF A FINITE GROUP

N.S. Kosenok, I.V. Blisnets

Francisk Skorina Gomel State University

Аннотация. В данной работе: G – конечная группа; $\sigma = \{\sigma_i \mid i \in I\}$ – некоторое разбиение множества всех простых чисел $\mathbb{P}$; $\Pi \subseteq \sigma$; $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \varnothing\}$ (n – целое число) и $\sigma(G) = \sigma(|G|)$. Группа G называется: (i) σ-примарной, если G является σ_i-группой для некоторого $i \in I$; (ii) σ-нильпотентной, если G – прямое произведение σ-примарных групп; П-группой, если $\sigma(G) \subseteq \Pi$. Подгруппа A конечной группы G называется: (i) σ-субнормальной в G, если в G существует цепь подгрупп $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ такая, что либо $A_{i-1} \trianglelefteq A_i$, либо $A_i / (A_{i-1})_{A_i}$ является σ-примарной группой для всех $i = 1, \ldots, t$; (ii) холловской П-подгруппой G, если A является П-группой и $\sigma(|G:A|) \cap \Pi = \varnothing$. Мы говорим, что подгруппа H группы G является строго σ-субнормальной, если H^G / H_G является σ-нильпотентной группой. В данной работе мы доказываем, что множество всех строго σ-субнормальных подгрупп, перестановочных с холловой П-подгруппой конечной группы G, образует подрешётку решётки всех подгрупп $L(G)$ группы G.

Ключевые слова: *конечная группа*, *решетка подгрупп*, *группа операторов*, *подрешетка решетки*, *холлова* П-*подгруппа*.

Для цитирования: *Косенок*, *Н.С.* Об одном классе подрешеток решетки подгрупп конечной группы / Н.С. Косенок, И.В. Близнец // Проблемы физики, математики и техники. – 2025. – № 4 (65). – С. 72–74. – DOI: https://doi.org/10.54341/20778708_2025_4_65_72. – EDN: KEQNCF

Abstract. In this paper: G is a finite group; $\sigma = \{\sigma_i \mid i \in I\}$ is some partition of the set of all primes $\mathbb{P}$; $\Pi \subseteq \sigma$; $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \varnothing\}$ (n is an integer) and $\sigma(G) = \sigma(|G|)$. A group G is said to be: (i) σ-primary provided G is a σ_i-group for some $i \in I$; (ii) σ-nilpotent if G is the direct product of σ-primary groups; a Π-group if $\sigma(G) \subseteq \Pi$. A subgroup A of a finte group G is said to be: (i) σ-subnormal in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; (ii) a Hall Π-subgroup of G if A is a Π-group and $\sigma(|G:A|) \cap \Pi = \varnothing$. We say that a subgroup H of G is strongly σ-subnormal if H^G / H_G is σ-nilpotent. In this paper, we prove that the set of all strongly σ-subnormal subgroups which permute with a Hall Π-subgroup of a finite group G forms a sublattice of the lattice of all subgroups $L(G)$ of G.

Keywords: *finite group*, *lattice of subgroups*, *operator group*, *sublattice of a lattice*, *Hall* Π-*subgroup*.

For citation: *Kosenok*, *N.S.* On a class of sublattices of the subgroup lattice of a finite group / N.S. Kosenok, I.V. Blisnets // Problems of Physics, Mathematics and Technics. – 2025. – № 4 (65). – P. 72–74. – DOI: https://doi.org/10.54341/20778708_2025_4_65_72. – EDN: KEQNCF

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes and $\sigma = \{\sigma_i \mid i \in I\}$ is some partition of $\mathbb{P}$; $\Pi \subseteq \sigma$ and $\Pi' = \sigma \setminus \Pi$.

If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G; $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \varnothing\}$ and $\sigma(G) = \sigma(|G|)$ [1]–[4].

A σ-*property* of a group [1]–[4] is understood to be any of its properties that depends on σ and which does not imply any restrictions on σ.

Before continuing, let us recall some of the most important concepts of the theory of σ-properties of a group.

A group G is said to be [1]–[4]:

(i) σ-*primary* if G is a σ_i-group for some $i \in I$;

(ii) σ-*nilpotent* if G is the direct product of σ-primary groups;

A subgroup A of G is said to be [1]–[4]:

(i) σ-*subnormal* in G if there is a subgroup chain

$$A = A_0 \le A_1 \le \cdots \le A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$;

(ii) σ-*permutable* in G provided G is σ-*full*, that is, G has a Hall σ_i-subgroup for all $i \in I$ and A permutes with all Hall σ_i-subgroups of G for all i.

We use H^G to denote the *normal closure* of the subgroup H in G (so H^G is the intersection of all normal subgroups of G containing H), and H_G is the *core* of H in G, that is, the subgroup of H generated by all normal subgroups of G contained in H.

Let us recall that a subgroup H of G is *strongly* σ-*subnormal in G* [6] if H^G / H_G is σ-nilpotent.

If $\sigma(H) \subseteq \Pi$, then H is called a Π-subgroup of G. A Π-subgroup H of G is called a *Hall* Π-*subgroup* of G if $\sigma(|G:H|) \cap \Pi = \varnothing$.

In this paper, we prove the following result.

***Theorem* 0.1.** *Let G be a group. If H is a Hall* Π-*subgroup of G, then the set of all strongly* σ-*subnormal subgroups of G which permute with H forms a sublattice in* $\mathcal{L}(G)$.

Taking in Theorem 0.1 $H = G$, we get from this theorem the following two results.

***Corollary* 0.2** (A.N. Skiba [6]). *The set of all strongly* σ-*subnormal subgroups of G forms a sublattice in* $\mathcal{L}(G)$.

***Corollary* 0.3.** *Let G be a group. If H is a Hall* Π-*subgroup of G, then the set of all strongly* σ-*subnormal subgroups of G which permute with H forms a sublattice in* $\mathcal{L}(G)$.

Let us recall that G is said to be: (i) a D_π-group if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E; (ii) a σ-*full group of Sylow type* [2] if every subgroup E of G is a D_{σ_i}-group for every $\sigma_i \in \sigma(E)$.

In view of [8, Theorem 1.2.14], every Sylow permutable subgroup of G is strongly subnormal in G. On the other hand, if G is a σ-full group of Sylow type, then every σ-pemutable subgroup is strongly σ-subnormal in G by Theorem B in [2]. Therefore, since the intersection of any set of sublattices of a lattice is a sublattice of this lattice, we also get from Theorem 0.1 the following two known results.

***Corollary* 0.4** (Kegel [9]). *The set of all Sylow permutable subgroups of G forms a sublattice in* $\mathcal{L}(G)$.

***Corollary* 0.5** (A.N. Skiba [2]). *If G ia a* σ-*full group of Sylow type, then the set of all* σ-*permutable subgroups of G forms a sublattice in* $\mathcal{L}(G)$.

1 Lemmas used

***Lemma* 1.1** (A.N. Skiba [2]). *The class* $\mathfrak{N}_\sigma$, *of all* σ-*nilpotent groups, is a hereditary Fitting formation.*

***Lemma* 1.2** [5, Ch. A, Proposition 1.6]. *Let* A, B *and* H *be subgroups of G. If* $AH = HA$ *and* $BH = HB$, *then* $\langle A, B\rangle H = H\langle A, B\rangle$.

We use $O^{\Pi}(G)$ to denote the subgroup of G generated by all its Π'-subgroups, where $\Pi' = \sigma \setminus \Pi$; $O_{\Pi}(G)$ is the product of all normal Π-subgroups of G.

***Lemma* 1.3** (A.N. Skiba [2, Lemma 2.6]). *If A is* σ-*subnormal in G and* $\sigma(|G:A|) \subseteq \Pi$ *-number, then* $O^{\Pi}(A) = O^{\Pi}(G)$.

2 Proof of Theorem 0.1

Proof. Let us assume that this theorem is false and let G be a counterexample of minimal order.

Let $\mathcal{L}$ be the set of all strongly σ-subnormal subgroups L of G which permute with H.

Let $A, B \in \mathcal{L}$ and let $K = \langle A, B\rangle$, $V = A \cap B$.

First we show that K is strongly σ-subnormal in G. By hypothesis, A^G / A_G is σ-nilpotent. Therefore, in view of the isomorphisms

$$A^G(A_G B_G)/A_G B_G \simeq A^G/(A^G \cap A_G B_G) =$$
$$= A^G / A_G(A^G \cap B_G) \simeq$$
$$\simeq (A^G/A_G)/(A_G(A^G \cap B_G)/A_A),$$

we get that

$$A^G(A_G B_G)/A_G B_G \in \mathfrak{N}_\sigma$$

since the class $\mathfrak{N}_\sigma$ is closed under taking homomorphic images by Lemma 1.1.

Similarly, we can get that

$$B^G(A_G B_G)/A_G B_G \in \mathfrak{N}_\sigma.$$

Moreover,

$$A^G B^G / A_G B_G =$$
$$= (A^G(A_G B_G)/A_G B_G)(B^G(A_G B_G)/A_G B_G)$$

and so, we have

$$A^G B^G / A_G B_G \in \mathfrak{N}_\sigma$$

since the class $\mathfrak{N}_\sigma$ is a Fitting formation by Lemma 1.1.

Next note that $\langle A, B\rangle^G = A^G B^G$ and $A_G B_G \le \langle A, B\rangle_G$. Therefore we get that

$$\langle A, B\rangle^G / \langle A, B\rangle_G \in \mathfrak{N}_\sigma$$

since the class $\mathfrak{N}_\sigma$ is closed under taking homomorphic images by Lemma 1.1. Hence $\langle A, B\rangle$ is strongly σ-subnormal in G.

Moreover, in view of Lemma 1.2,

$$\langle A,B\rangle H = H\langle A,B\rangle$$

since $AH = HA$ and $BH = HB$ by the choice of A and B. Therefore $K \in \mathcal{L}$.

Now we show that $V \in \mathcal{L}$. First note that

$$(A\cap B)_G = A_G \cap B_G.$$

On the other hand, from the isomorphism

$$(A^G \cap B^G)/(A_G \cap B^G) =$$
$$= (A^G \cap B^G)/(A_G \cap B^G \cap A^G) \simeq$$
$$\simeq A_G(B^G \cap A^G)/A_G \le A^G/A_G$$

we get that

$$(A^G \cap B^G)/(A_G \cap B^G) \in \mathfrak{N}_\sigma$$

since the class $\mathfrak{N}_\sigma$ is closed under taking normal subgroup by Lemma 1.1. Similarly, we get that

$$(B^G \cap A^G)/(B_G \cap A^G) \in \mathfrak{N}_\sigma.$$

But then we get that

$$(A^G \cap B^G)/(A_G \cap B_G) \in \mathfrak{N}_\sigma$$

since the class $\mathfrak{N}_\sigma$ is a formation by Lemma 1.1.

It is also clear that

$$(A\cap B)^G \le A^G \cap B^G.$$

Therefore we get that

$$(A\cap B)^G/(A\cap B)_G \in \mathfrak{N}_\sigma.$$

Therefore $A\cap B$ is strongly σ-subnormal in G.

Finally, we show that $V = A\cap B$ is permutable with H. Let us assume that this is false. Then G is not a Π-group, since otherwise we have $H = G$ and so

$$G = (A\cap B)H = H(A\cap B).$$

First, let us assume that $R := (A\cap B)_G \ne 1$. Then

$$(A/R)^G/(A/R)_G = (A^G/R)/(A_G/R) \simeq A^G/A_G$$

is σ-nilpotent, so A/R is strongly σ-subnormal in G/R. Similarly, B/R is strongly σ-subnormal in G/R. It is also clear that HR/R is a Hall Π-subgroup of G/R and A/R and B/R permute with HR/R, so the choice of G implies that

$$((A\cap B)/R)(HR/R) =$$
$$= ((A/R)\cap(B/R))(HR/R) =$$
$$= (HR/R)((A/R)\cap(B/R)) =$$
$$= (HR/R)((A\cap B)/R).$$

But then

$$(A\cap B)H = (A\cap B)HR = HR(A\cap B) = H(A\cap B),$$

which is a contradiction.

Thus, $(A\cap B)_G = 1$, so $(A\cap B)^G$ is σ-nilpotent and hence $(A\cap B)^G = V\times W$, where W is a Hall Π-subgroup of $(A\cap B)^G$. Then $W \le H$. It is also clear that $A\cap B = L\times K$, where K is a Hall Π-subgroup of $A\cap B$ and that $K \le H$. Moreover,

$$L = O^\Pi(A\cap B) = O_{\Pi'}(A\cap B).$$

Now we show that $H \le N_G(L)$. Indeed, we have $H \le N_G(O^\Pi(A))$ and $H \le N_G(O^\Pi(B))$ by Lemma 1.3, so $H \le N_G(O^\Pi(A)\cap O^\Pi(B))$.

Now observe that $O^\Pi(A)\cap O^\Pi(B)$ is normal in $A\cap B$ and from

$$(A\cap B)/(A\cap O^\Pi(A)\cap B) \simeq (A\cap B)O^\Pi(A)/O^\Pi(A)$$

and

$$(A\cap B)/(B\cap O^\Pi(B)\cap A) \simeq (A\cap B)O^\Pi(B)/O^\Pi(B)$$

we get that

$$(A\cap B)/(O^\Pi(A)\cap O^\Pi(B)) =$$
$$= (A\cap B)/((A\cap O^\Pi(A)\cap B)\cap(B\cap O^\Pi(B)\cap A))$$

is a Π-group. Hence

$$L = O_{\Pi'}(A\cap B) = O_{\Pi'}(O^\Pi(A)\cap O^\Pi(B)),$$

so $H \le N_G(L)$.

Since $A\cap B = L\times K$, where $K \le H$ and $H \le N_G(L)$, we have

$$(A\cap B)H = (L\times K)H = LH = HL =$$
$$= H(L\times K) = H(A\cap B),$$

a contradiction. Therefore $V \in \mathcal{L}$, so $\mathcal{L}$ is a sublattice of the lattice $\mathcal{L}(G)$. □

REFERENCES

1. *Skiba, A.N.* On σ-properties of finite groups I / A.N. Skiba // Problems of Physics, Mathematics and Technics. – 2014. – № 4 (21). –P. 89–96.

2. *Skiba, A.N.* On σ-subnormal and σ-permutable subgroups of finite groups / A.N. Skiba // J. Algebra. – 2015. – № 436. – P. 1–16.

3. *Skiba, A.N.* A generalization of a Hall theorem / A.N. Skiba // J. Algebra Appl. – 2015. – Vol. 15, № 4. – P. 21–36.

4. *Skiba, A.N.* On Some Results in the Theory of Finite Partially Soluble Groups / A.N. Skiba // Commun. Math. Stat. – 2016. – № 4. – P. 281–309.

5. *Doerk, K.* Finite soluble groups / K. Doerk, T. Hawkes. – Berlin – New York: Walter de Gruyter, 1992.

6. *Skiba, A.N.* On sublattices of the subgroup lattice defined by formation Fitting sets / A.N. Skiba // J. Algebra. – 2020. – № 550. – P. 69–85.

7. *Shemetkov, L.A.* Formations of Algebraic Systems / L.A. Shemetkov, A.N. Skiba // Moscow: Nauka, 1989.

8. *Ballester-Bolinches, A.* Products of Finite Groups / A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad. – Berlin – New York: Walter de Gruyter, 2010.

9. *Kegel, O.H.* Sylow-Gruppen and Subnormalteilerendlicher Gruppen / O.H. Kegel // Math. Z. – 1962. – № 78. – P. 205–221.

The article was submitted 12.09.2025.

Информация об авторах

Косенок Николай Сергеевич – к.ф.-м.н., доцент

Близнец Игорь Васильевич – к.ф.-м.н., доцент