МАТЕМАТИКА

УДК 517.956.35 DOI: https://doi.org/10.54341/20778708_2024_2_59_73
EDN: LGNECC

ОБ ОТСУТСТВИИ, НЕЕДИНСТВЕННОСТИ И РАЗРУШЕНИИ КЛАССИЧЕСКИХ РЕШЕНИЙ СМЕШАННЫХ ЗАДАЧ ДЛЯ ТЕЛЕГРАФНОГО УРАВНЕНИЯ С НЕЛИНЕЙНЫМ ПОТЕНЦИАЛОМ

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ON THE ABSENCE, NON-UNIQUENESS, AND BLOW-UP OF CLASSICAL SOLUTIONS OF MIXED PROBLEMS FOR THE TELEGRAPH EQUATION WITH A NONLINEAR POTENTIAL

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Аннотация. Для телеграфного уравнения с нелинейным потенциалом, заданным в первом квадранте, рассматриваются первая и вторая смешанные задачи, для которых исследуются вопросы, связанные с отсутствием, неединственностью и разрушением классических решений.

Ключевые слова: *полулинейное волновое уравнение*, *смешанная задача*, *классическое решение*, *отсутствие решения*, *неединственность решения*, *разрушение решения*, *метод характеристик*, *энергетические методы*, *условия сопряжения*.

Для цитирования: *Корзюк*, *В.И.* Об отсутствии, неединственности и разрушении классических решений смешанных задач для телеграфного уравнения с нелинейным потенциалом / В.И. Корзюк, Я.В. Рудько // Проблемы физики, математики и техники. – 2024. – № 2 (59). – С. 73–78. – DOI: https://doi.org/10.54341/20778708_2024_2_59_73. – EDN: LGNECC

Abstract. For the telegraph equation with a nonlinear potential given in the first quadrant, we consider the first and the second mixed problem, for which we study issues related to the absence, non-uniqueness, and blow-up of classical solutions.

Keywords: *semilinear wave equation*, *mixed problem*, *classical solution*, *absence of solution*, *non-uniqueness of solution*, *blow-up of solution*, *method of characteristics*, *energy methods*, *matching conditions.*

For citation: *Korzyuk*, *V.I.* On the absence, non-uniqueness, and blow-up of classical solutions of mixed problems for the telegraph equation with a nonlinear potential / V.I. Korzyuk, J.V. Rudzko // Problems of Physics, Mathematics and Technics. – 2024. – № 2 (59). – P. 73–78. – DOI: https://doi.org/10.54341/20778708_2024_2_59_73 (in Russian). – EDN: LGNECC

Introduction

Continuous media are described mainly by nonlinear partial differential equations. The choice of linear or nonlinear equations for describing a medium depends on the role played by nonlinear effects and is determined by the specific physical situation. For example, when describing the propagation of laser pulses, it is necessary to take into account the dependence of the refractive index of the medium on the electromagnetic field intensity. The linearization of nonlinear equations of mathematical physics does not always lead to meaningful results. It may turn out that the linearized equations apply to the physical process in question only for some finite time. Moreover, from the viewpoint of physics, it is often “essentially nonlinear” solutions, qualitatively different from the solutions of linear equations, that are extremely important for nonlinear equations of mathematical physics. These can be stationary solutions of the soliton type, localized in one or several dimensions, or solutions of the wave collapse type, which describe the spontaneous concentration of energy in small regions of space. Stationary solutions of hydrodynamic equations are also essentially nonlinear [1].

Nonlinear partial differential equations are difficult to study: almost no general techniques exist that work for all such equations, and usually each individual equation has to be studied as a separate problem. A fundamental question for any partial differential equation is the existence and uniqueness of a solution for given boundary conditions. For nonlinear equations these questions are in general very hard [2].

For the existence and uniqueness of global classical solutions of mixed problems for *linear* hyperbolic[1] PDEs, it is necessary and sufficient to satisfy: 1) the smoothness conditions; 2) the matching conditions.

[1] For non-hyperbolic equations, as a rule, it is necessary to specify additional growth conditions on the initial data, e.g., see [5].

However, this is not enough for *non-linear* hyperbolic PDEs. As a rule, we should impose additional conditions on the nonlinearity to establish the existence and uniqueness of global classical solutions for non-linear PDEs. For example, we can take one of the following conditions:

1. Constraint on the growth of the nonlinearity. It includes the Lipschitz condition, the Carathéodory condition, and other similar conditions [6, 7].

2. Sign condition, e.g., [13, p. 670–671].

3. Matching of nonlinearities and initial data, i.e., some classes of nonlinearities require small initial data, e.g., [26–30].

We also note that a global classical solution can exist even if the nonlinearity is a non-differentiable function. In addition, the problem of the existence and uniqueness of global classical solutions is more complicated than for weak ones because, in particular, classical solutions do not admit singularities.

This paper deals with the question of the absence, non-uniqueness, and blow-up of global and local classical solutions of the telegraph equation with a nonlinear potential. This article contains our preliminarily announced results [3], [4].

1 Statement of the problem

In the domain $Q=(0,\infty)\times\mathbb{R}$ of two independent variables $(t,x)\in(0,\infty)\times\mathbb{R}\subset\mathbb{R}^2$, consider the one-dimensional nonlinear equation

$$\left(\partial_t^2-a^2\partial_x^2\right)u(t,x)-f\left(t,x,u(t,x)\right)=F(t,x),\quad (t,x)\in Q, \tag{1.1}$$

where $a\in(0,\infty)$, F is a function given on the set $\overline{Q}$, and f is a function given on the set $\overline{Q}\times\mathbb{R}$. Equation (1.1) is equipped with the initial condition

$$u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x),\ x\in[0,\infty), \tag{1.2}$$

and the boundary condition

$$B[u](t,0)=\mu(t),\ t\in[0,\infty). \tag{1.3}$$

where ϕ, ψ, and μ are functions given on the half-line $(0,\infty)$ and B is some operator (it can have various forms, but in the present work we assume that $B=I$ or $B=\partial_x$, where I is the identity operator).

2 Nonexistence of solutions

2.1 Inhomogeneous matching conditions

***Assertion* 2.1.** *Assume* $B=I$. *If the homogeneous matching conditions*

$$\mu(0)=\varphi(0),\ \mu'(0)=\psi(0),$$

$$\mu''(0)=\frac{1}{2}\left(f\left(0,0,\varphi(0)\right)+f\left(0,0,\mu(0)\right)\right)+$$
$$+F(0,0)+a^2\varphi(0),$$

fail for given functions f, ϕ, ψ, μ, *and F*, *then*, *for any smoothness of these functions*, *the first mixed problem* (1.1)–(1.3) *does not have a classical solution defined on* $\overline{Q}$.

The **proof** follows from Theorem 1 from [6], [7].

***Assertion* 2.2.** *Assume* $B=\partial_x$. *If the homogeneous matching conditions*

$$\mu(0)=\varphi'(0),\quad \mu'(0)=\psi'(0),$$

fail for given functions f, ϕ, ψ, μ, *and F*, *then*, *for any smoothness of these functions*, *the second mixed problem* (1.1)–(1.3) *does not have a classical solution defined on* $\overline{Q}$.

The **proof** can be carried out by the method of characteristics by analogy with that of the preceding assertion.

***Remark* 2.1.** Violation of the matching conditions specified in Assertions 1, 2 is not critical, since in this case we can consider the problem with the conjugation conditions on the characteristic $x-at=0$ and seek its classical solutions. This question is discussed in more detail in [8]–[11] (for linear equations) and [6], [7] (for nonlinear equations).

2.2 Negative energy

In this subsection we will use the energy method to show some conditions under which the mixed problem (1.1)–(1.3) does not have a global classical solution. This approach was developed by H. Levine, who used it to study a wide class of initial value problems for second-order nonlinear wave equations [12]. This method introduces appropriate integral quantities depending on the time variable *t*, for which we can then derive differential inequalities, involving convexity, that lead to contradictions [13]. The general idea of the energy method is that the initial data with negative energy means that there is no global solution. For some classes of problems, the negative energy actually means that the initial data are quite "large" [14].

However, this method does not cover all possible cases of the mixed problem (1.1)–(1.3); this approach requires, for example, compactly supported initial data. So, we impose the following restrictions on the nonlinearity, the right side of the equation, the initial data, and the boundary data of the problems.

***Condition* 2.1.** *The functions F and* μ *are identically equal to zero*, *the function f has the form* $f(t,x,z)=-g(z)$, *where* $g(0)=0$, *and the smoothness conditions* $\varphi\in C_c^2([0,\infty))$, $\psi\in C_c^2([0,\infty))$, *and* $g\in C^1(\mathbb{R})$, *are satisfied.*

Under condition 1, we introduce the notation

$$G(z)=\int_0^z g(\xi)d\xi,\quad z\in\mathbb{R}.$$

and define the energy of a solution *u* of the problem (1.1)–(1.3)

$$E:[0,\infty)\ni t\mapsto E(t)=$$
$$=\int_0^\infty\left(\frac{1}{2}\left(\left(\frac{\partial u}{\partial t}(t,x)\right)^2+a^2\left(\frac{\partial u}{\partial x}(t,x)\right)^2\right)+\right.$$
$$\left.+G\left(u(t,x)\right)\right)dx\in\mathbb{R}.$$

***Lemma* 2.1.** *Assume that Condition* 2.1 *holds. Then classical solution u of the problem* (1.1)–(1.3) *has compact support in space for each time t, i. e., the function* $x \mapsto u(t,x)$ *has compact support in space for each* $t \in [0,\infty)$.

Proof. Let x^* be a real number such that $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi) \subseteq [0, x^*]$. It means that for any real number $x > x^*$ the equality $\varphi(x) = \psi(x) = 0$ holds. For real numbers t_0 and x_0 such that $x_0 - at_0 > x^*$, define

$$e(t) = \int_{x_0 - a(t_0 - t)}^{x_0 + a(t_0 - t)} \left(\frac{1}{2} \left(\left(\frac{\partial u}{\partial t}(t,x) \right)^2 + a^2 \left(\frac{\partial u}{\partial x}(t,x) \right)^2 + \left(u(t,x) \right)^2 \right) \right) dx, \quad t \in [0, t_0].$$

Then

$$e'(t) = \int_{x_0 - a(t_0 - t)}^{x_0 + a(t_0 - t)} \left(\left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial t \partial x} \right)(t,x) + u(t,x) \frac{\partial u}{\partial t}(t,x) \right) dx -$$
$$- a\mathcal{L}(t, x_0 + a(t_0 - t)) - a\mathcal{L}(t, x_0 - a(t_0 - t)) =$$
$$= \int_{x_0 - a(t_0 - t)}^{x_0 + a(t_0 - t)} \left(\frac{\partial u}{\partial t}(t,x) \left(\frac{\partial^2 u}{\partial t^2}(t,x) - a^2 \frac{\partial^2 u}{\partial x^2}(t,x) + u(t,x) \right) \right) dx -$$
$$- a\mathcal{L}(t, x_0 + a(t_0 - t)) - a\mathcal{L}(t, x_0 - a(t_0 - t)) +$$
$$+ a^2 \mathcal{B}(t, x_0 + a(t_0 - t)) - a^2 \mathcal{B}(t, x_0 - a(t_0 - t)), \quad t \in [0, t_0].$$

where

$$\mathcal{L}(t,x) := \frac{1}{2} \left(\left(\frac{\partial u}{\partial t}(t,x) \right)^2 + a^2 \left(\frac{\partial u}{\partial x}(t,x) \right)^2 \right) + \left(u(t,x) \right)^2,$$

$$\mathcal{B}(t,x) := \frac{\partial u}{\partial x}(t,x) \frac{\partial u}{\partial t}(t,x).$$

By virtue of the Cauchy – Bunyakovsky – Schwarz inequality

$$\pm \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right)(t,x) \le \frac{a}{2} \left(\frac{\partial u}{\partial x}(t,x) \right)^2 + \frac{1}{2a} \left(\frac{\partial u}{\partial x}(t,x) \right)^2,$$

we have

$$e'(t) \le \int_{x_0 - a(t_0 - t)}^{x_0 + a(t_0 - t)} \left(\frac{\partial u}{\partial t}(t,x) \left(u(t,x) - g(u(t,x)) \right) \right) dx,$$
$$t \in [0, t_0].$$

Since $g(0) = 0$ and $u \in C^2(\Omega)$, where

$$\Omega = \operatorname{Conv}\{(t_0, x_0), (0, x_0 - at_0), (0, x_0 + at_0)\},$$
$$g(u(t,x)) \le C|u(t,x)|, \quad (t,x) \in \Omega$$

for some constant C depending upon $\|u\|_{L^\infty(\Omega)}$. By analogy with [13, p. 662], we conclude $e'(t) \le Ce(t)$. As $e(0) = 0$, Grönwall's inequality [13, p. 708] implies $e \equiv 0$. Therefore $u \equiv 0$ within the set Ω. By virtue of the arbitrariness of t_0 and x_0 such that $x_0 - at_0 > x^*$, $u(t,x) = 0$ for any $x > x^* + at$ and $t \in [0,\infty)$. The proof of the lemma is complete.

***Lemma* 2.2** (Conservation of energy). *Assume that Condition* 2.1 *holds and u is a classical solution of the problem* (1.1)–(1.3). *Then* $t \mapsto E(t)$ *is constant.*

Proof. We calculate

$$E'(t) = \int_0^\infty \left(\left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial t \partial x} \right)(t,x) + g(u(t,x)) \frac{\partial u}{\partial t}(t,x) \right) dx =$$
$$= a^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \Bigg|_{(t,x=0)}^{(t,x=\infty)} + \int_0^\infty \left(\frac{\partial u}{\partial t}(t,x) \left(\frac{\partial^2 u}{\partial t^2}(t,x) - a^2 \frac{\partial^2 u}{\partial x^2}(t,x) + g(u(t,x)) \right) \right) dx = 0.$$

The integration by parts in this proof is valid, since u has compact support in space for each time, according to Lemma 2.1.

***Assertion* 2.3.** Assume that Condition 2.1 holds, for some constant $\lambda > 2$ we have the inequality

$$zg(z) \le \lambda G(z), \quad z \in \mathbb{R}.$$

Suppose also that the energy is negative

$$E(0) = \int_0^\infty \left(\frac{1}{2} \left(\left(\varphi'(x) \right)^2 + a^2 \left(\psi(x) \right)^2 \right) + G(\varphi(x)) \right) dx < 0.$$

Then the problem (1.1)–(1.3) does not have a classical solution defined on $\bar{Q}$.

The **proof** repeats word for word the proof of Theorem 1 in [13, §12.5.1].

Note that similar results have been obtained for slightly other equations [23]–[25].

3 Nonuniqueness of solutions

In this section, we consider the second mixed problem (1.1)–(1.3) in the following case

$$f(t,x,z) := z^\alpha, \ 0 < \alpha < 1, \ F(t,x) = 0, \quad \varphi = \psi = \mu \equiv 0, \ B = \partial_x. \tag{3.1}$$

It is easy to see that the mixed problem (1.1)–(1.3) has the trivial solution $u \equiv 0$. To find non-trivial solutions of the problem (1.1)–(1.3), consider the ansatz

$$u(t,x) = u(t) = \beta t^\gamma, \quad (t,x) \in \bar{Q}, \tag{3.2}$$

where β and γ are some real numbers. Substituting ansatz (3.2) into Eq. (1.1), we obtain the relation

$$\beta(\gamma - 1)\gamma t^{\gamma - 2} = \beta^\alpha t^{\alpha\gamma},$$

which leads to the system of equations

$$\gamma - 2 = \gamma\alpha, \quad \beta(\gamma - 1)\gamma = \beta^\alpha,$$

which has the solution

$$\beta = 2^{1/(\alpha-1)}\left(\alpha - 3 + 4/(\alpha+1)\right)^{1/(1-\alpha)},$$
$$\gamma = 2/(1-\alpha). \quad (3.3)$$

Substituting (3.3) into (3.2), we get the function

$$u_p(t,x) = 2^{\frac{1}{\alpha-1}}\left(\frac{\alpha+1}{\alpha^2-2\alpha+1}\right)^{\frac{1}{\alpha-1}} t^{\frac{2}{1-\alpha}}. \quad (3.4)$$

It is easy to see that the function (3.4) satisfies the initial (1.2) and boundary conditions (1.3). Thus, we have constructed one nontrivial solution of the problem (1.1)–(1.3), (3.1), which is determined by the formula (3.4). Moreover, it can be easily shown that the 'glued' [15] solution

$$u_{p;s}(t,x) = \begin{cases} 0, & t \in [0,s), \\ u_p(t-s,x), & t \in [s,+\infty), \end{cases}$$

with parameter $s > 0$ also satisfies the problem (1.1)–(1.3), (3.1). Thus, we have constructed the infinite set of nontrivial classical solutions of the problem (1.1)–(1.3), (3.1).

We note that in the problem (1.1)–(1.3), (3.1) the nonlinearity $u \mapsto -u^\alpha$ is not a differentiable function on the set $\mathbb{R}$. It is the fact that makes the construction of a unique local classical solution of the problem (1.1)–(1.3), (3.1) impossible, because in the case of continuously differentiable nonlinearity, we can build a local classical solution (but the matching conditions have to be satisfied). We can do this using the methods proposed in the works [6], [7], [13], [16], [17]. We state the result as the following assertion.

Assertion 3.1. The second mixed problem (1.1) – (1.3) has an infinite number of global classical solutions defined on $\bar{Q}$ and no unique local classical solution.

The **proof** follows from the above argument.

We can use this approach to prove the non-uniqueness of the classical solution of other problems [18], [19].

4 Blow-up of solutions

4.1 Positive nonlinearities

We consider the second mixed problem (1.1)–(1.3) in the following case

$$f(t,x,z) := g(z),\ g \ge 0,\ F(t,x) = 0,$$
$$\varphi = \psi = \mu \equiv 0,\ B = \partial_x. \quad (4.1)$$

To find non-trivial solutions of the problem (1.1) – (1.3), (4.1), consider the ansatz

$$u(t,x) = u(t),\ (t,x) \in \bar{Q}. \quad (4.2)$$

It leads to the Cauchy problem for an ODE

$$u''(t) = g(u(t)),\ u(0) = 0,\ u'(0) = 0,$$

which can have the blow-up of non-trivial solutions [20].

For example, we can set

$$g(z) = z^2 + 1, \quad z \in \mathbb{R}, \quad (4.3)$$

and check the conditions specified in the paper [20]. It gives a result that a classical solution of the problem (1.1)–(1.3), (4.1), (4.3) has blow-up in a finite time. We formulate the result of this section as the following statement.

***Assertion* 4.1.** Assume $g \in C^2([0,\infty))$, $g(x) \ge 0$ for all $x \ge 0$, and the integral $\int_0^\infty \left(\int_0^\xi g(s)ds\right)^{-1/2} d\xi$ converges. The second mixed problem (1.1)–(1.3), (3.1) has a nontrivial solution, which blow-ups in a finite time, i. e., there exists $0 < T_* < +\infty$ such that $\lim_{t \to T_*-0} u(t,x) = +\infty$.

The **proof** follows from the paper [20] and the above argument.

4.2 Power law nonlinearities

We consider the second mixed problem (1.1)–(1.3) in the following case

$$f(t,x,z) := \lambda|z|^\alpha,\ \lambda > 0,\ \alpha > 1,\ F(t,x) = 0,$$
$$\varphi \equiv \varphi_0 > 0,\ \psi \equiv \psi_0 > 0,\ \mu \equiv 0,\ B = \partial_x. \quad (4.4)$$

To find non-trivial solutions of the problem (1.1)–(1.3), (4.4), again consider the ansatz (4.2). It leads to the Cauchy problem for an ODE

$$u''(t) = \lambda|u(t)|^\alpha,\ u(0) = \varphi_0,\ u'(0) = \psi_0. \quad (4.5)$$

According to the Peano existence theorem there exists at least one solution to the problem (4.5) on the interval $[0,T)$. On the other hand, the problem (4.5) satisfies conditions (F1) – (F4) of Theorem 1.1 of the paper [21]. Therefore, the right maximum existence interval $[0,T^*)$ of the solution u of the problem (4.5) is finite (i. e., $T^* < +\infty$), and

$$\lim_{t \to T_*-0} u(t,x) := \lim_{t \to T_*-0} u(t) = +\infty,$$
$$\lim_{t \to T_*-0} \partial_t u(t,x) := \lim_{t \to T_*-0} u'(t) = +\infty.$$

We can say the same for the nonlinearity of the form $f(t,x,z) := \lambda z|z|^{\alpha-1}$, where $\lambda > 0$ and $\alpha > 1$. We can formulate a more general statement.

***Assertion* 4.1.** Assume the function $h : \mathbb{R}^2 \ni (t,z) \mapsto h(t,z) \in \mathbb{R}$ satisfies the following conditions:

1) For any bounded subset $\Omega \subset \mathbb{R}$ the function $h|_{\mathbb{R}\times\Omega}$ is bounded.

2) There exist $\beta > 1$, $c_0 > 0$, and $c_1 > 0$ such that $h(t,z) \ge c_0 z^\beta - c_1$ for all $t \in \mathbb{R}$ and $z \ge 0$.

Let M_0 be a number such that $h(t,z) \ge \varepsilon_0 > 0$ for all $t \in \mathbb{R}$ and $z \ge M_0$. The second mixed problem (1.1) – (1.3) has a solution that blows up in a finite time (i. e., there exists $0 < T_* < +\infty$ such that $\lim_{t \to T_*-0} \partial_t u(t,x) = +\infty$), if the following conditions are satisfied:

$$F(t,x) = 0,\ B = \partial_x,\ \varphi \equiv \varphi_0 = \text{const} \ge M_0,$$
$$\psi \equiv \psi_0 = \text{const} > 0,\ \mu \equiv 0,\ f(t,x,u) = h(t,u).$$

Additionally, if there exists a continuous nondecreasing function $\alpha : [0,+\infty) \mapsto [0,+\infty)$ such that

$h(t,z) \le \alpha(|z|)$ for all $t \in \mathbb{R}$ and $z \in \mathbb{R}$, then we have $\lim_{t \to T_* - 0} u(t,x) = +\infty$.

Proof. We will look for a solution u having the form (9). It leads to the following Cauchy problem:

$$u''(t) = h(t, u(t)),\ u(0) = \varphi_0,\ u'(0) = \psi_0.$$

Using [21, Theorem 1.1] we conclude that there exists $0 < T_* < +\infty$ such that $\lim_{t \to T_* - 0} u'(t) = +\infty$, i. e., $\lim_{t \to T_* - 0} \partial_t u(t,x) := \lim_{t \to T_* - 0} u'(t) = +\infty$. From [21, Theorem 1.1] it also follows $\lim_{t \to T_* - 0} u'(t) = +\infty$, i. e., $\lim_{t \to T_* - 0} u(t,x) = \lim_{t \to T_* - 0} u(t) = +\infty$, if there exists a non-decreasing function $\alpha \in C([0,+\infty))$ such that $h(t,z) \le \alpha(|z|)$ for all $(t,z) \in \mathbb{R}^2$.

To construct an explicit example, consider the following case of the second mixed problem: $f(t,x,z) := 2z^3$, $F \equiv 0$, $\varphi \equiv 1$, $\psi \equiv -1$, and $\mu \equiv 0$. In which the problem has an explicit classical solution $u_*^1(t,x) = (1-t)^{-1}$ on the set $[0,1) \times [0,\infty)$ (we can strictly justify this by direct verification). We have

$$\lim_{t \to 1-0} u_*^1(t,x) = +\infty,\ \lim_{t \to 1-0} \partial_t u_*^1(t,x) = +\infty,$$

$$u_*^1 \in C^2([0,1) \times [0,\infty)).$$

However, this solution can be analytically extended to the entire complex plane (with respect to the argument *t*), except for the line $t = 1$, where there will be a first-order pole.

Conclusions

In this article, we have shown that the fulfillment of the smoothness conditions and the matching conditions is not enough for the existence of a global classical solution of boundary value problems for the telegraph equation with a nonlinear potential, unlike the linear telegraph equation. Also, these conditions do not lead to the uniqueness of solutions. For the existence and uniqueness of a global classical solution, the nonlinearity of the equation has to satisfy some additional conditions, e. g., the Lipschitz condition. But on the other hand, the Lipschitz condition is not necessary for the existence of a unique global classical solution [22].

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The article was submitted 20.12.2023.

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