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ДИРАКОПОДОБНЫЕ УРАВНЕНИЯ И ОБОБЩЕННЫЕ МАЙОРАНОВСКИЕ ПОЛЯ, ВНУТРЕННЯЯ СИММЕТРИЯ

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DIRAC LIKE EQUATIONS AND GENERALIZED MAJORANA FIELDS, INTRINSIC SYMMETRIES

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Аннотация. Для многокомпонентного матричного уравнения $(\Gamma_\mu \partial_\mu + m)\psi = 0$ вводится понятие внутренней симметрии. Эти симметрии должны сохранять форму уравнения и соответствующий лагранжиан должен быть инвариантен относительно преобразования симметрии. Накладывается дополнительное требование: преобразования симметрии должны сохранять майорановскую природу полей. Это означает, что если функция Ψ_A является действительной (мнимой) частью волновой функции, то после преобразования функция остается действительной (мнимой). Исследованы многокомпонентные поля Майораны, которые могут быть связаны с одним, двумя, тремя и четырьмя полями Дирака, как массивными, так и безмассовыми. Установлены группы преобразований симметрии для этих полей.

Ключевые слова: *обобщенные дираковские и майорановские поля*, *лагранжев формализм*, *внутренняя симметрия.*

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Abstract. We start with the the multicomponent matrix equation $(\Gamma_\mu \partial_\mu + m)\psi = 0$, and introduce the concept of the intrinsic symmetry. These symmetries should preserve the form of the basic equation. The relevant Lagrangian should be invariant under the intrinsic symmetry transformation. We will impose one additional requirement on symmetry transformations: such transformations should preserve the Majorana nature of the fields. This means that if the function Ψ_A is real (imaginary) part of the wave function, then after symmetry transformation the function remains real (imaginary). The situation for massless field $\Gamma_\mu \partial_\mu \psi = 0$ is substantially different. The Lagrangian invariance with respect to intrinsic symmetry transformation for massless case coincide with that for massive case. The main accent will be done on multicomponent Majorana fields, which can be related to one, two, three and four Dirac fields.

Keywords: *generalized Dirac and Majorana fields*, *Lagrangian formalism*, *intrinsic symmetry.*

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Introduction

The theory of relativistic wave equations is the base for description of the elementary particles and their interaction. It started with the investigations of P.A.M. Dirac [1], W. Pauli [2], and M. Fierz [3]. These studies were proceeded by H.J. Bhabha [4] and Harish – Chandra [5]. They proposed for description of particles to apply the first order equations in matrix form

$$(\Gamma_\mu \partial_\mu + m)\Psi = 0, \qquad (0.1)$$

where Ψ stands for wave functions, Γ_μ designates square matrices, *m* is the mass parameter.

In this field, the investigation by I.M. Gelfand and A.M. Yaglom [6] in which the general method for constructing the wave equations in matrix form (0.1) for particles with any sets of spin and mass states was developed was very important. Substantial contribution in this theory was done by F.I. Fedorov et. all [7], [8]. Important contribution in studying the algebras of the matrices Γ_μ and development of the methods of calculation was done by L.A. Shelepin [9]. Also significant contribution was done by V.I. Fuschich and A.G. Nikitin [10], [11]. They proved existence of invariance for many physical equations on the base of non Lie-like symmetries.

There exists a special way for describing the intrinsic degrees of freedom and additional characteristics of the particles, it is based on the use of extended sets of representations of the Lorentz group. The known example is the Dirac-Kähler system referring to the particle with two spin states $(s = 0,1)$ and degeneration in intrinsic parities. Firstly, the Dirac-Käahler equation was formulated in tensor form by C.G. Darwin [12]. For describing the electron in external Coulomb field he proposed to apply the complicated tensor system of equations. In this way, he derived the energy spectrum in presence of external Coulomb field which coincides with that in the Dirac theory. Later on, this Darwin equation was rediscovered by many researchers. The mostly known is the paper by E. Kähler [13], where the formalism of differential forms was used. In the papers of V.I. Strazhev with coauthors [14], [19], this system was studied in detail within the conventional theory of relativistic wave equations.

Intrinsic symmetries for massless Dirac equation were considered many years ago by W. Pauli [20] and F. Gursey [21]. The main goal of the present paper is to generalize their approach to the cases of 2, 3, 4 Dirac fields, both massive and massless, the accent will be given to the case of Majorana particles related to 1, 2, 3, 4 bispinor fields.

1 Basic Definitions

Let us start with the matrix equation and corresponding Lagrangian

$$(\Gamma_\mu \partial_\mu + m)\psi = 0, \quad L = -\psi^{+}\eta(\Gamma_\mu \partial_\mu + m)\psi. \quad (1.1)$$

Under the intrinsic symmetry transformations we mean linear transformations $\Psi'_A = Q_{AB}\Psi_B$ which obey a number of conditions. They should preserve the form of equation (1.1), this leads to

$$(\Gamma_\mu \partial_\mu + m)Q\psi = 0 \Rightarrow (\Gamma_\mu \partial_\mu + m)\psi' = 0, \quad [Q, \Gamma_\mu]_- = 0. \quad (1.2)$$

Lagrangian (1.1) should be invariant under the transformation Q, this requirement provides us with the following restriction

$$Q^{+}\eta Q = \eta. \quad (1.3)$$

We will impose one additional requirement on symmetry transformations. Such transformations should preserve the Majorana nature of the field. This means that if the function Ψ_A is real (imaginary) part of the complete wave function, then after symmetry transformation the function $\Psi'_A = Q_{AB}\Psi_B$ remains real (imaginary). Henceforth, this requirement is called the Majorana condition. Now let us specify the massless case

$$\Gamma_\mu \partial_\mu \psi = 0. \quad (1.4)$$

The requirement of invariance for that equation leads to two alternative restrictions

$$\Gamma_\mu \partial_\mu Q_1 \psi = 0 \Rightarrow [Q_1, \Gamma_\mu]_- = 0, \quad -\Gamma_\mu \partial_\mu Q_2 \psi = 0 \Rightarrow [Q_2, \Gamma_\mu]_+ = 0. \quad (1.5)$$

Additional requirement of Lagrangian invariance also leads to two possibilities. One is $L' = L, \left[Q_1, \Gamma_\mu\right]_- = 0;$ it reduces to yet known constraint (1.3), which arose in the massive case. The other possibility is as follows

$$L' = -L, \left[Q_2, \Gamma_\mu\right]_+ = 0;$$

whence we obtain $Q_2^{+}\eta\Gamma_\mu Q_2 = -\eta\Gamma_\mu$. Keeping in mind the relation $\left[Q_2, \Gamma_\mu\right]_+ = 0,$ we conclude that the last relation is equivalent to the known restriction (1.3). Thus, the Lagrangian invariance with respect to intrinsic symmetry transformation both for massive and massless cases assumes one and the same constraint (1.3).

For infinitesimal one-parametric intrinsic symmetry transformation $Q = 1 + \omega J,$ relation (1.3) takes on the simple form

$$(\omega J)^{+}\eta = -\eta\, \omega J. \quad (1.6)$$

2 One Dirac Field

Let us consider one Dirac equation for a particle with nonzero mass $(\gamma_\mu \partial_\mu + m)\psi = 0,$ where ψ transforms as a bispinor; we used the metric with imaginary unit, $x_\mu = (\vec{x}, ict)$. Below we will apply Majorana representation to the Dirac matrices [22]:

$$\gamma_1 = \sigma_1 \otimes \sigma_1, \gamma_2 = \sigma_3 \otimes I_2, \quad \gamma_3 = \sigma_1 \otimes \sigma_3, \gamma_4 = \sigma_1 \otimes \sigma_2, \quad (2.1)$$

σ_i designate the Pauli matrices. Allowing for identities $\gamma_i^{*} = \gamma_i, \gamma_4^{*} = -\gamma_4, \partial_i^{*} = \partial_i, \partial_4^{*} = -\partial_4,$ we get

$$\left(\gamma_1\partial_1 + \gamma_2\partial_2 + \gamma_3\partial_3 + \gamma_4\partial_4 + m\right)\psi^{*} = 0. \quad (2.2)$$

Summing and subtracting two last equations, we obtain $(\Gamma_\mu \partial_\mu + m)\Psi = 0,$ where the 8-component wave function Ψ has the structure

$$\Psi = (\psi^r, \psi^i), \psi^r = \frac{1}{\sqrt{2}}(\psi + \psi^{*}), \psi^i = \frac{1}{\sqrt{2}}(\psi - \psi^{*}), \quad (2.3)$$

the matrices Γ_μ are defined by the formula $\Gamma_\mu = I_2 \otimes \gamma_\mu$. In this Majorana basis, the most general form of transformation Q (1.2) is

$$Q = \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} \otimes I_4 = q \otimes I_4, \quad q_{mn} \in C; \quad (2.4)$$

8-dimensional symmetry transformations are decomposed into the linear combinations

$$Q = \omega_0 I + \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3, \quad (2.5)$$

the matrices J_k satisfy the commutation relations for $su(2)$:

$$J_1 = \frac{\sigma_1}{2} \otimes I_4, J_2 = \frac{i\sigma_2}{2} \otimes I_4, J_3 = \frac{\sigma_3}{2} \otimes I_4, \quad [J_i, J_j]_- = i\varepsilon_{ijk} J_k. \quad (2.6)$$

The above Majorana condition leads to the following restrictions on parameters: ω_1 is imaginary, and ω_2, ω_3 are real, below we will apply the notations $\omega_1 = i\Omega_1, \omega_2 = \Omega_2, \omega_3 = \Omega_3$. The determinant of the Q equals

$$\det Q = \left(-1-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)\times \times\left(1-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)\times \times\left(-i-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)\times \times\left(i-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)+1;$$

because the total multiplier at Q has no physical meaning, we set $\det Q = +1$, so obtaining

$$\left(-1-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)\times \times\left(1-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)\times \times\left(-i-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)\times \times\left(i-\omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2\right)=0.$$

whence we get two alternative possibilities

$$\Omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2=1, \quad \Omega_1{}^2-\omega_2{}^2-\omega_3{}^2+\omega_0{}^2=-1. \tag{2.7}$$

The existence of Lagrangian formulation (1.2) leads to additional restrictions: the symmetry transformation may include only one generator J_1:

$$Q = \omega_0 I + i\Omega_1 J_1; \tag{2.8}$$

correspondingly relations (2.7) take on the form

$$\Omega_1{}^2+\omega_0{}^2=1, \quad \Omega_1{}^2+\omega_0{}^2=-1. \tag{2.9}$$

It is readily verified that the Majorana condition forbids the second variant in (2.9). The finite transformation has the structure

$$\psi^{r'}=\omega_0\psi^r+i\Omega_1\psi^i, \ \psi^{i'}=\omega_0\psi^i+i\Omega_1\psi^r, \quad \Omega_1{}^2+\omega_0{}^2=1. \tag{2.10}$$

Real and imaginary parts get entangled by this transformation, however the spiting into real and imaginary parts is not destroyed. Transformations (2.10) make up the Abelian group $U(1)$.

In fact, this model can be easily reduced to the form when we may speak about two 8-dimensional Majorana fields, real and imaginary. Indeed, let it be $i\psi_i = \overline{\phi}_r, i\psi_r = \overline{\phi}_i$, then (2.9) are re-written as follows

$$\psi_{r'} = \omega_0\psi_r + \Omega_1\overline{\phi}_r, \quad \psi_i' = \omega_0\psi_i + \Omega_1\overline{\phi}_i.$$

3 The System of Two Dirac Fields

Let us consider the system of two Dirac fields

$$(\gamma_\mu\partial_\mu + m)\psi_1 = 0, \quad (\gamma_\mu\partial_\mu + m)\psi_2 = 0, \tag{3.1}$$

where ψ_1, ψ_2 stand for two bispinors, as in the above we apply the Dirac matrices to Majorana basis. Further we derive the standard matrix form of the 16-component equation

$$(\Gamma_\mu\partial_\mu + m)\Psi = 0, \ \Psi = (\psi_1^r, \psi_2^r; \psi_1^i, \psi_2^i), \quad \Gamma_\mu = I_4 \otimes \gamma_\mu. \tag{3.2}$$

The most general form of the relevant symmetry transformations should have the structure $Q = q \otimes I_4$, where q is a certain 4×4 matrix. This matrix can be decomposed into the complete set $I_4, \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_\mu\gamma_\nu$, where indices in $\gamma_\mu\gamma_\nu$ take the the values $\{23, 31, 12, 14, 24, 34\}$. Therefore, the transformations Q may be presented with the help of 16 basic elements

$$I_{16}, J^\mu = \gamma_\mu \otimes I_4, J^5 = \gamma_5 \otimes I_4, \quad J^{\mu 5} = i\gamma_\mu\gamma_5 \otimes I_4, J^{\mu\nu} = i\gamma_\mu\gamma_\nu \otimes I_4; \tag{3.3}$$

expressions for $J^{\mu 5}$ and $J^{\mu\nu}$ are multiplied by imaginary unit in order to have corresponding generators Hermitian. Let us numerate the generators as follows

$$J^\mu \to J_1 ... J_4, \ J^5 \to J_5, \quad J^{\mu 5} \to J_6 ... J_9, \ J^{\mu\nu} \to J_{10} ... J_{15}. \tag{3.4}$$

Applying the Majorana requirement to 1-parametric transformations

$$Q = 1 + \omega_s J_s, \ s = 1,...,15, \text{(no summing in } s) \tag{3.5}$$

we get additional restrictions on parameters:

– imaginary $\omega_1, \omega_3, \omega_7, \omega_9, \omega_{11}, \omega_{14}$;

– real $\omega_2, \omega_4, \omega_5, \omega_6, \omega_8, \omega_{10}, \omega_{12}, \omega_{13}, \omega_{15}$.

From the Lagrangian invariance we get 15 restrictions on generators

$$(\omega_s J_s)^+ \eta = -\eta\omega_s J_s, \ s = 1,...,15, \ \eta = I_4 \otimes \gamma_4. \tag{3.6}$$

The direct verification of equations (3.6) with the use of explicit expressions for generators shows that only 6 generators satisfy these constraints $J_1, J_3, J_7, J_9, J_{11}, J_{14}$. Thus, the Lagrangian is invariant only under 1-parametric transformations with generators

$$J_1 = \gamma_1 \otimes I_4, J_3 = \gamma_3 \otimes I_4, \quad J_{11} = i\gamma_3\gamma_1 \otimes I_4, J_7 = i\gamma_2\gamma_5 \otimes I_4, \quad J_9 = i\gamma_4\gamma_5 \otimes I_4, J_{14} = i\gamma_2\gamma_4 \otimes I_4. \tag{3.7}$$

These generators lead to finite transformations with the structure

$$\begin{vmatrix} R_1 & iR_2 \\ iR_3 & R_4 \end{vmatrix} \begin{vmatrix} \Psi_+ \\ i\Psi_- \end{vmatrix}, \tag{3.8}$$

where R_1, R_2, R_3, R_4 are real 8×8 matrices, and Ψ_+, Ψ_- are real 8-dimensional columns. These transformations entangle 8 real and 8 imaginary components, however the splitting into real and imaginary part is not destroyed. It is readily verified that two triples of generators

$$S_1 = \frac{1}{2}J_7, S_2 = \frac{1}{2}J_9, S_3 = \frac{1}{2}J_{14}; \quad S_1' = \frac{1}{2}J_1, S_2' = \frac{1}{2}J_3, S_3' = \frac{1}{2}J_{11}, \tag{3.9}$$

obey the Lie algebra $su(2)$: $[S_i, S_j]_- = iS_k\varepsilon_{ijk}$ and $[S_i', S_j']_- = iS_k'\varepsilon_{ijk}$. These two sets commute with each other, $[S_i, S_j']_- = 0$. In other words, these transformations make up a 6-parametric group with the structure $SU(2) \otimes SU(2)$.

4 The System of Three Dirac Fields

Let us consider the system of three Dirac fields

$$(\gamma_\mu \partial_\mu + m)\psi_i = 0, i = 1,2,3, \quad (4.1)$$

where ψ_1, ψ_2, ψ_3 are bispinors. We obtain the standard matrix form of the equation

$$(\Gamma_\mu \partial_\mu + m)\Psi = 0, \ \Gamma_\mu = I_6 \otimes \gamma_\mu, \quad \Psi = (\psi_1^r, \psi_2^r, \psi_3^r, \psi_1^i, \psi_2^i, \psi_3^i). \quad (4.2)$$

Intrinsic symmetry transformations Q are presented by complex 24×24 matrices, which commute with the matrices Γ_μ. In Majorana basis, the most general structure of the matrix Q is $Q = q \otimes I_4$, where q is a complex 6×6 matrix. This matrix q can be decomposed in the linear combination of the basic matrices

$$I_6, \quad \sigma_i \otimes I_3, \quad I_2 \otimes \alpha_A, \quad \sigma_i \otimes \alpha_A; \quad (4.3)$$

where α_A stand for generators of the group $SU(3)$, $A = 1 \div 8$.

Let us take 8 Hermitian generators α_A for the group $SU(3)$ as follows [23]:

$$\alpha_1 = e^{11} - e^{33}, \ \alpha_2 = e^{22} - e^{33}, \ \alpha_3 = e^{23} + e^{32},$$
$$\alpha_4 = e^{13} + e^{31}, \alpha_5 = e^{12} + e^{21}, \alpha_6 = -i(e^{23} - e^{32}), \quad (4.4)$$
$$\alpha_7 = -i(e^{31} - e^{13}), \ \alpha_8 = -i(e^{12} - e^{21}),$$

where e_{ij} stand for the elements of the complete matrix algebra. Their explicit form is

$$\alpha_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \ \alpha_2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \ \alpha_3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix},$$
$$\alpha_4 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \ \alpha_5 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad (4.5)$$
$$\alpha_6 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{vmatrix}, \ \alpha_7 = \begin{vmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{vmatrix}, \ \alpha_8 = \begin{vmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

They relate to Okubo matrices [24] in the following way

$$a_1^1 = \frac{1}{3}\begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \ a_1^2 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \ a_1^3 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$
$$a_2^1 = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \ a_2^2 = \frac{1}{3}\begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{vmatrix},$$
$$a_2^3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \ a_3^1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad (4.6)$$
$$a_3^2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \ a_3^3 = \frac{1}{3}\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix}.$$

We can easily derive the following relations between these two sets

$$\alpha_1 = 2a_1^1 + a_2^2, \ \alpha_2 = a_1^1 + 2a_2^2, \ \alpha_3 = a_2^3 + a_3^2,$$
$$\alpha_4 = a_1^3 + a_3^1, \ \alpha_5 = a_1^2 + a_2^1, \ \alpha_6 = -i(a_2^3 - a_3^2),$$
$$\alpha_7 = -i(a_1^3 - a_3^1), \ \alpha_8 = -i(a_1^2 - a_2^1).$$

In application of the group $SU(3)$, the Gell-Mann matrices are commonly used [23], they are related to the above matrices α_i (4.4) by the formulas

$$\lambda_1 = \frac{1}{2}\alpha_5, \ \lambda_2 = \frac{1}{2}\alpha_8, \ \lambda_3 = \frac{1}{2}(\alpha_1 - \alpha_2),$$
$$\lambda_4 = \frac{1}{2}\alpha_4, \ \lambda_5 = \frac{1}{2}\alpha_7, \ \lambda_6 = \frac{1}{2}\alpha_3, \quad (4.8)$$
$$\lambda_7 = \frac{1}{2}\alpha_6, \ \lambda_8 = \frac{1}{2\sqrt{3}}(\alpha_1 + \alpha_2).$$

Let us turn back to the study of the symmetries Q for a 24-component field. The relevant transformations are determined by 35 Hermitian generators; it is convenient to numerate them

$$J_1 ... J_3 \to (\sigma_i \otimes I_3) \otimes I_4,$$
$$J_4 ... J_{11} \to (I_2 \otimes \alpha_A) \otimes I_4, \quad (4.9)$$
$$J_{12} ... J_{35} \to (\sigma_i \otimes \alpha_A) \otimes I_4.$$

It should be noted that only generators J_1, J_2, J_3 have quadratic minimal polynomial, the remaining 32 generators have the cubic minimal polynomial: $3 \to \lambda^2 = 1; 32 \to \lambda^3 = \lambda.$ Minimal polynomials for generators based on Gell-Mann 3×3 matrices have more complex structure:

$$J_1, J_2, J_3 \to \lambda^2 = 1; \ J_{11} \to \lambda^2 + \frac{\sqrt{3}}{3}\lambda = \frac{2}{3};$$
$$J_{19}, J_{27}, J_{35} \to \lambda^4 - \frac{5}{3}\lambda^2 = \frac{4}{9}; \quad (4.10)$$

for 28 remaining generators the minimal polynomials are cubic $\lambda^3 = \lambda$. Below we will apply the generators (4.9).

The Majorana condition for 1-parametric transformations leads to the following constraints for 35 parameters ω:

– real

$$\omega_2, \omega_3, \omega_4, \omega_6, \omega_7, \omega_9,$$
$$\omega_{11}, \omega_{13}, \omega_{16}, \omega_{31}, \omega_{33}, \omega_{35}; \quad (4.11)$$
$$\omega_{18}, \omega_{20}, \omega_{22}, \omega_{23}, \omega_{25}, \omega_{27}, \omega_{28}, \omega_{30},$$

– imaginary

$$\omega_1, \omega_5, \omega_8, \omega_{10}, \omega_{12}, \omega_{14}, \omega_{15}, \omega_{17},$$
$$\omega_{19}, \omega_{21}, \omega_{24}, \omega_{26}, \omega_{29}, \omega_{32}, \omega_{34}. \quad (4.12)$$

The Lagrangian requirement (1.6) is satisfied only for imaginary parameters (4.12).

Thus, the intrinsic symmetry transformations are determined by the following 15 generators

$$J_1 = (\sigma_1 \otimes I_3) \otimes I_4, \ J_9 = (I_2 \otimes \alpha_6) \otimes I_4,$$
$$J_{10} = (I_2 \otimes \alpha_7) \otimes I_4, \ J_{11} = (I_2 \otimes \alpha_8) \otimes I_4,$$
$$J_{12} = (\sigma_1 \otimes \alpha_1) \otimes I_4, \ J_{13} = (\sigma_1 \otimes \alpha_2) \otimes I_4,$$

$$J_{14}=(\sigma_1\otimes\alpha_3)\otimes I_4,\ J_{15}=(\sigma_1\otimes\alpha_4)\otimes I_4,$$
$$J_{16}=(\sigma_1\otimes\alpha_5)\otimes I_4,\ J_{25}=(\sigma_2\otimes\alpha_6)\otimes I_4,$$
$$J_{26}=(\sigma_2\otimes\alpha_7)\otimes I_4,\ J_{27}=(\sigma_2\otimes\alpha_8)\otimes I_4,$$
$$J_{33}=(\sigma_3\otimes\alpha_6)\otimes I_4,\ J_{34}=(\sigma_3\otimes\alpha_7)\otimes I_4,$$
$$J_{35}=(\sigma_3\otimes\alpha_8)\otimes I_4. \quad (4.13)$$

Only the generator J_1 has a quadratic minimal polynomial, the 14 remaining ones have a cubic minimal polynomial. The study of commutators for generators shows that there exist two triples of generators which make up subgroups isomorphic to $su(2)$:

$$\frac{1}{2}(J_9,J_{13},J_{14})=(S_1,S_2,S_3),$$
$$\frac{1}{2}(J_{10},J_{12},J_{15})=(S_1',S_2',S_3'). \quad (4.14)$$

All the generators in sets (4.14) have cubic minimal polynomial; besides, the generators from different triples commute with each other. Recall that these triples are realized on the matrices of dimension 24×24.

Let us write down the structure of the finite 1-parametric transformations relation to generators (4.13). The finite 1-parametric transformations for generators with minimal polynomial are

$$U=1+i\sin\alpha\lambda+(\cos\alpha-1)\lambda^2; \quad (4.15)$$

for the case of a quadratic polynomial we get

$$U=\cos\alpha-i\sin\alpha\lambda.$$

Because all 15 one-parametric transformations are symmetries, we can conclude that all products of them will provide us with symmetries as well.

5 The System of 4 Dirac Fields

Let us consider the system of 4 Dirac fields

$$(\gamma_\mu\partial_\mu+m)\psi_i=0,\quad (i=1,2,3,4), \quad (5.1)$$

whence we get the standard matrix equation

$$(\Gamma_\mu\partial_\mu+m)\Psi=0,\ \Gamma_\mu=I_8\otimes\gamma_\mu,$$
$$\Psi=(\psi_1^r,\psi_2^r,\psi_3^r,\psi_4^r,\psi_1^i,\psi_2^i\psi_3^i\psi_4^i). \quad (5.2)$$

Transformations of intrinsic symmetry Q are determined by complex 32×32 matrices, they should commute with the matrices Γ_μ. In Majorana basis the most general form of Q is as follows $Q=q\otimes I_4$, where q stands for a complex 8×8 matrix. It can be decomposed in the complete set of basic 8×8 matrices:

$$I_8,\gamma_\mu\otimes I_2,\gamma_5\otimes I_2,\gamma_\mu\gamma_5\otimes I_2,\gamma_\mu\gamma_\nu\otimes I_2,$$
$$\gamma_\mu\otimes\sigma_i,\gamma_5\otimes\sigma_i,\gamma_\mu\gamma_5\otimes\sigma_i,\gamma_\mu\gamma_\nu\otimes\sigma_i,I_4\otimes\sigma_i. \quad (5.3)$$

The symmetry transformations for a 32-component field are determined by 63 generators; let us list them as shown below

$$J_\mu\to J_1...J_4\to(\gamma_\mu\otimes I_2)\otimes I_4,$$
$$J_5\to J_5\to(\gamma_5\otimes I_2)\otimes I_4,$$
$$J_{\mu5}\to J_6...J_9\to i(\gamma_\mu\gamma_5\otimes I_2)\otimes I_4,$$
$$J_{[\mu\nu]}\to J_{10}...J_{15}\to i(\gamma_\mu\gamma_\nu\otimes I_2)\otimes I_4,$$
$$J_{\mu i}\to J_{16}...J_{27}\to(\gamma_\mu\otimes\sigma_i)\otimes I_4,$$
$$J_{5i}\to J_{28}...J_{30}\to(\gamma_5\otimes\sigma_i)\otimes I_4, \quad (5.4)$$
$$J_{[\mu5i]}\to J_{31}...J_{42}\to i(\gamma_\mu\gamma_5\otimes\sigma_i)\otimes I_4,$$
$$J_{[\mu\nu]i}\to J_{43}...J_{60}\to i(\gamma_\mu\gamma_\nu\otimes\sigma_i)\otimes I_4,$$
$$J_{4i}\to J_{61}...J_{63}\to(I_4\otimes\sigma_i)\otimes I_4.$$

All generators are Hermitian, and have a quadratic minimal polynomial, $J_{...}^2=I$. The Majorana condition for 1-parametric transformations leads to the constrains on 63 parameters ω:

– real 35

$$\omega_2,\omega_4,\omega_5,\omega_6,\omega_8,\omega_{10},\omega_{12},\omega_{13},\omega_{15},$$
$$\omega_{17},\omega_{19},\omega_{21},\omega_{23},\omega_{25},\omega_{27},\omega_{28},\omega_{30},$$
$$\omega_{31},\omega_{33},\omega_{35},\omega_{37},\omega_{39},\omega_{41},\omega_{43},\omega_{45},\omega_{47},$$
$$\omega_{49},\omega_{51},\omega_{52},\omega_{54},\omega_{56},\omega_{58},\omega_{60},\omega_{61},\omega_{63}; \quad (5.5)$$

– imaginary 28

$$\omega_1,\omega_3,\omega_7,\omega_9,\omega_{11},\omega_{14},\omega_{16},\omega_{18},\omega_{20},$$
$$\omega_{22},\omega_{24},\omega_{26},\omega_{29},\omega_{32},\omega_{34},\omega_{36},\omega_{38}, \quad (5.6)$$
$$\omega_{40},\omega_{42},\omega_{44},\omega_{46},\omega_{48},\omega_{50},\omega_{53},\omega_{55},\omega_{57},\omega_{59},\omega_{62}.$$

The Lagrangian formulation (1.6) of the theory is possible only for 28 one-parametric transformations with imaginary ω (5.6). Thus, the intrinsic symmetry transformations are determined by the 28 generators

$$J_1=(\gamma_1\otimes I_2)\otimes I_4, J_3=(\gamma_3\otimes I_2)\otimes I_4,$$
$$J_7=i(\gamma_2\gamma_5\otimes I_2)\otimes I_4, J_9=i(\gamma_4\gamma_5\otimes I_2)\otimes I_4,$$
$$J_{11}=i(\gamma_3\gamma_1\otimes I_2)\otimes I_4, J_{14}=i(\gamma_2\gamma_4\otimes I_2)\otimes I_4,$$
$$J_{16}=(\gamma_1\otimes\sigma_1)\otimes I_4, J_{18}=(\gamma_1\otimes\sigma_3)\otimes I_4,$$
$$J_{20}=(\gamma_2\otimes\sigma_2)\otimes I_4, J_{22}=(\gamma_3\otimes\sigma_1)\otimes I_4, \quad (5.7)$$
$$J_{24}=(\gamma_3\otimes\sigma_3)\otimes I_4, J_{26}=(\gamma_4\otimes\sigma_2)\otimes I_4,$$
$$J_{29}=(\gamma_5\otimes\sigma_2)\otimes I_4, J_{32}=i(\gamma_1\gamma_5\otimes\sigma_2)\otimes I_4,$$
$$J_{34}=i(\gamma_2\gamma_5\otimes\sigma_1)\otimes I_4, J_{36}=i(\gamma_2\gamma_5\otimes\sigma_3)\otimes I_4,$$
$$J_{38}=i(\gamma_3\gamma_5\otimes\sigma_2)\otimes I_4, J_{40}=i(\gamma_4\gamma_5\otimes\sigma_1)\otimes I_4,$$
$$J_{42}=i(\gamma_4\gamma_5\otimes\sigma_3)\otimes I_4, J_{44}=i(\gamma_2\gamma_3\otimes\sigma_2)\otimes I_4,$$
$$J_{46}=i(\gamma_3\gamma_1\otimes\sigma_1)\otimes I_4, J_{48}=i(\gamma_3\gamma_1\otimes\sigma_3)\otimes I_4,$$
$$J_{50}=i(\gamma_1\gamma_2\otimes\sigma_2)\otimes I_4, J_{53}=i(\gamma_1\gamma_4\otimes\sigma_2)\otimes I_4,$$
$$J_{55}=i(\gamma_2\gamma_4\otimes\sigma_1)\otimes I_4, J_{57}=i(\gamma_2\gamma_4\otimes\sigma_3)\otimes I_4,$$
$$J_{59}=i(\gamma_3\gamma_4\otimes\sigma_2)\otimes I_4, J_{62}=(I_4\otimes\sigma_2)\otimes I_4.$$

All the generators have dimension 32×32, and can be presented with the use of blocks of dimension 8×8. The study of the structure of these generators permits us to make the following conclusions.

1. Among the generators (5.6) one can separate 56 triples, each of them obeys the commutative relations of the Lie group $su(2)$. For instance the triples, (J_7,J_{70},J_{29}), (J_7,J_{32},J_{50}), (J_{70},J_{40},J_{55}) and so on.

2. For each of 56 triples there exist 10 other triples which commute with the generators from the first triple. For instance, the triple (J_{16},J_{36},J_{59}) commutes with the following 10 concomitant triples

$$(J_1, J_{20}, J_{50}), (J_1, J_{24}, J_{48}), (J_1, J_{29}, J_{32}),$$
$$(J_7, J_{70}, J_{29}), (J_7, J_{32}, J_{50}),$$
$$(J_{70}, J_{40}, J_{55}), (J_{20}, J_{40}, J_{48}), (J_{24}, J_{32}, J_{55}),$$
$$(J_{24}, J_{40}, J_{50}), (J_{29}, J_{48}, J_{55}). \quad (5.8)$$

The generators from the basic triple do not enter concomitant 10 triples

$$[J_{\text{basic}}, J^A_{\text{concomit}}]_- = 0, \; A = 1 \div 10;$$
$$J_{\text{basic}} \cap J^A_{\text{concomit}} = 0. \quad (5.9)$$

In other words, each triple generates 10 subgroups with the structure $su(2) \otimes su(2)$.

6 The System of One Massless Dirac Field

Let us consider one Dirac equation with zero mass $\gamma_\mu \partial_\mu \psi = 0$, it may be presented in matrix form $\Gamma_\mu \partial_\mu \Psi = 0$, where Ψ is an 8-component wave function (2.3). Because the field under consideration is massless, the intrinsic symmetry transformations may commute or anticommute with the basic matrices $[Q_1, \Gamma_\mu]_- = 0$, $[Q_2, \Gamma_\mu]_+ = 0$. The first condition was analyzed in the above. So we are to study only the second condition. The structure of symmetries Q_2 should be as follows $Q_2 = q_2 \otimes \gamma_5$, where q_2 stands for an arbitrary complex matrix 2×2. Because the matrix q_2 can be decomposed in the set of $I_2, \sigma_1, \sigma_2, \sigma_3$, the symmetries Q_2 are determined by 4 elements (for massless cases, we will designate generators by symbol L):

$$L_1 = \sigma_1 \otimes \gamma_5, \; L_2 = \sigma_2 \otimes \gamma_5,$$
$$L_3 = \sigma_3 \otimes \gamma_5, \; L_0 = I_2 \otimes \gamma_5. \quad (6.1)$$

The Majorana condition leads to the following restrictions on parameters of 1-parametric transformations $Q_2 = 1 + \Omega L$: Ω_1 is real, Ω_2, Ω_3, Ω_0 are imaginary. The existence of the Lagrangian formulation (1.6) is possible only for one generator $L_1 = \sigma_1 \otimes \gamma_5$. Let us recall that the first symmetry transformation Q_1 leads to the following result

$$J_1 = \sigma_1 \otimes I_4, \; (\omega_1 \text{ is imaginary}). \quad (6.2)$$

We can see that transformations corresponding to J_1 and L_1 are substantially different. Let us consider the finite transformations Q_1 and Q_2:

$$Q_1 = a_0 I_8 + i a_1 J_1, \; Q_2 = b_0 I_8 + b_1 L_1; \quad (6.3)$$

a_i, b_i are real. For these symmetries, the Lagrangian condition $Q^+ \eta Q = \eta$ leads to restrictions

$$a_0^2 + a_1^2 = 1, \; b_0^2 - b_1^2 = 1. \quad (6.4)$$

Evidently, the product of Q_1 and Q_2 also is a symmetry transformation

$$Q = Q_1 Q_2 = Q_2 Q_1 =$$
$$= a_0 b_0 I_8 + a_0 b_1 L_1 + i b_0 a_1 J_1 + i a_1 b_1 J_1 L_1, \quad (6.5)$$

where $J_1 L_1 = L_1 J_1 = I_2 \otimes \gamma_5$. It is readily proved that the Lagrangian condition for the transformation (6.5) leads to restriction

$$(a_0^2 + a_1^2)(b_0^2 - b_1^2) = 1. \quad (6.6)$$

Imposing the proper normalization, we rewrite the formulas (6.3) as follows

$$Q_1 = \cos\alpha I_8 + i \sin\alpha J_1,$$
$$Q_2 = \cosh\beta I_8 + \sinh\beta L_1. \quad (6.7)$$

7 The System of Two Massless Fields

Let us consider the system of two equations

$$\gamma_\mu \partial_\mu \psi_1 = 0, \gamma_\mu \partial_\mu \psi_2 = 0 \Rightarrow \Gamma_\mu \partial_\mu \Psi = 0,$$
$$\Psi = (\psi_1^r, \psi_2^r, \psi_1^i, \psi_2^i). \quad (7.1)$$

The intrinsic symmetry transformations obey the commutation or anticommutation relations $[Q_1, \Gamma_\mu]_- = 0, [Q_2, \Gamma_\mu]_+ = 0$. The study of the commutation condition was performed in the above. Below we shall analyze the anticommutation condition. The structure of relevant matrix Q_2 should be $Q_2 = q \otimes \gamma_5$. The matrix $q_{4\times4}$ can be decomposed into the set of 16 matrices

$$I_4, \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \gamma_\mu, \gamma_\mu\gamma_5, \gamma_\mu\gamma_\nu.$$

Therefore the intrinsic symmetry transformations Q_2 may be defined with the help of 15 generators

$$L^\mu = \gamma_\mu \otimes \gamma_5, L^5 = \gamma_5 \otimes \gamma_5,$$
$$L^{\mu 5} = i\gamma_\mu\gamma_5 \otimes \gamma_5, L^{\mu\nu} = i\gamma_\mu\gamma_\nu \otimes \gamma_5. \quad (7.2)$$

Let us numerate them as follows

$$L^\mu \to L_1 ... L_4, L^5 \to L_5,$$
$$L^{\mu 5} \to L_6 ... L_9, L^{\mu\nu} \to L_{10} ... L_{15}. \quad (7.3)$$

For 1-parametric transformations $Q_2 = 1 + \Omega L$, obeying the Majorana condition, we find the following restrictions on parameters

– real $\Omega_1, \Omega_3, \Omega_7, \Omega_9, \Omega_{11}, \Omega_{14}$;

– imaginary $\Omega_2, \Omega_4, \Omega_5, \Omega_6, \Omega_8, \Omega_{10}, \Omega_{12}, \Omega_{13}, \Omega_{15}$.

The study of Lagrangian condition (1.6) shows that the appropriate are the generators corresponding to real-valued parameters

$$L_1 = \gamma_1 \otimes \gamma_5, L_3 = \gamma_3 \otimes \gamma_5, L_{11} = i\gamma_3\gamma_1 \otimes \gamma_5,$$
$$L_7 = i\gamma_2\gamma_5 \otimes \gamma_5, L_9 = i\gamma_4\gamma_5 \otimes \gamma_5, L_{14} = i\gamma_2\gamma_4 \otimes \gamma_5. \quad (7.4)$$

Let us study the Lagrangian condition for finite transformations Q_2 (1.3):

$$Q_2^+ \eta Q_2 = \eta, \; \eta = I_4 \otimes \gamma_4,$$
$$Q_2 = b_0 I_{16} + b_1 L_1 + b_2 L_3 + b_3 L_7 + b_4 L_9 + b_5 L_{11} + b_6 L_{14};$$

whence we find two solutions

1) $Q_2 = b_0 I_{16} + b_1 L_1 + b_2 L_3 + b_5 L_{11}$, $b_0^2 - b_1^2 - b_2^2 - b_5^2 = 1$;

2) $Q_2 = b_0 I_{16} + b_3 L_7 + b_4 L_9 + b_6 L_{14}$, $b_0^2 - b_3^2 - b_4^2 - b_6^2 = 1$; (7.5)

all parameters b_i are real-valued, so in parametric space the signature is $(+,-,-,-)$.

Similarly, we consider the Lagrangian condition for finite transformations Q_1 (1.3)

$$Q_1^{+}\eta Q_1 = \eta,\ \ \eta = I_4 \otimes \gamma_4,$$
$$Q_1 = a_0 I_{16} + ia_1 J_1 + ia_2 J_3 +$$
$$+ia_3 J_7 + ia_4 J_9 + ia_5 J_{11} + ia_6 J_{14},$$

whence we obtain two solutions

$$1)\ Q_1 = a_0 I_{16} + ia_1 J_1 + ia_2 J_3 + ia_5 J_{11},$$
$$a_0^2 + a_1^2 + a_2^2 + a_5^2 = 1;$$
$$2)\ Q_1 = a_0 I_{16} + ia_3 J_7 + ia_4 J_9 + ia_6 J_{14},$$
$$a_0^2 + a_3^2 + a_4^2 + a_6^2 = 1; \qquad (7.6)$$

all parameters a_i are real, in the parametric space we have the signature $(+,+,+,+)$. In relations (7.5) and (7.6) all the generators are Hermitian.

Let us change the notations for the generators

$$S_1 = \frac{1}{2}J_7,\ S_2 = \frac{1}{2}J_9,\ S_3 = \frac{1}{2}J_{14},$$
$$S_1' = \frac{1}{2}J_1,\ S_2' = \frac{1}{2}J_3,\ S_3' = \frac{1}{2}J_{11}; \qquad (7.7)$$
$$s_1 = \frac{1}{2}L_7,\ s_2 = \frac{1}{2}L_9,\ s_3 = \frac{1}{2}L_{14},$$
$$s_1' = \frac{1}{2}L_1,\ s_2' = \frac{1}{2}L_3,\ s_3' = \frac{1}{2}L_{11}.$$

Then for symmetries Q_1 we get more symmetrical formulas

$$1)\ Q_1 = a_0 I_{16} + ia_3 S_1 + ia_4 S_2 + ia_6 S_3,$$
$$[S_i, S_j]_- = -iS_k \varepsilon_{ijk}, \qquad (7.8)$$
$$2)\ Q_1 = a_0 I_{16} + ia_1 S_1' + ia_2 S_2' + ia_5 S_3',$$
$$[S_i', S_j']_- = -iS_k' \varepsilon_{ijk},$$

in (7.8) we can see two commuting 3-parametric groups with the structure $su(2)$, $[S_i, S_j']_- = 0$. For the case Q_2 we have

$$1)\ Q_2 = b_0 I_{16} + b_3 s_1 + b_4 s_2 + b_6 s_3,$$
$$2)\ Q_2 = b_0 I_{16} + b_1 s_1' + b_2 s_2' + b_5 s_3'. \qquad (7.9)$$

We can see that all four triples of the generators from symmetries Q_1 and Q_2 are mixed in the following way:

$$[s_i, s_j]_- = -i\varepsilon_{ijk} S_k,\ [s_i', s_j']_- = -i\varepsilon_{ijk} S_k',$$
$$[S_i, s_j]_- = -is_k \varepsilon_{ijk},\ [S_i', s_j']_- = -is_k' \varepsilon_{ijk}. \qquad (7.10)$$

Within the commutating relations (7.10) we can separate two 6-parametric subgroups:

– the first is

$$(a_0 I_{16} + ia_3 S_1 + ia_4 S_2 + ia_6 S_3)(b_0 I_{16} + b_3 s_1 + b_4 s_2 + b_6 s_3),$$
$$[S_i, S_j]_- = -iS_k \varepsilon_{ijk},\ [s_i, s_j]_- = -iS_k \varepsilon_{ijk}, \qquad (7.11)$$
$$[S_i, s_j]_- = -is_k \varepsilon_{ijk};$$

– the second is

$$(a_0 I_{16} + ia_1 S_1' + ia_2 S_2' + ia_5 S_3')(b_0 I_{16} + b_1 s_1' + b_2 s_2' + b_5 s_3'),$$
$$[S_i', S_j']_- = -iS_k' \varepsilon_{ijk},\ [s_i', s_j']_- = -iS_k' \varepsilon_{ijk}, \qquad (7.12)$$
$$[S_i', s_j']_- = -is_k' \varepsilon_{ijk}.$$

These groups are isomorphic to $SO(4)$ group (see in [25]). Thus, the complete symmetry group for 2 massless Dirac fields in Majorana approach is $\mathrm{SO}(4) \otimes \mathrm{SO}(4)$.

8 The System of Three Massless Fields

Let us consider the system of three Dirac fields

$$\gamma_\mu \partial_\mu \psi_i = 0\ (i = 1,2,3) \Rightarrow \Gamma_\mu \partial_\mu \Psi = 0,$$
$$\Psi = (\psi_1^r, \psi_2^r, \psi_3^r, \psi_1^i, \psi_2^i, \psi_3^i). \qquad (8.1)$$

For symmetry transformations, two alternative constraints may be imposed

$$[Q_1, \Gamma_\mu]_- = 0, \quad \text{or} \quad [Q_2, \Gamma_\mu]_+ = 0. \qquad (8.2)$$

The first restriction was analyzed in the above. Here we shall examine the second condition. The general structure of the transformations Q_2 may be as follows $Q_2 = q \otimes \gamma_5$, where q is an arbitrary complex 6×6 matrix. Any such matrix may be decomposed into the complete set

$$I_6,\ \sigma_i \otimes I_3,\ I_2 \otimes \alpha_A,\ \sigma_i \otimes \alpha_A, \qquad (8.3)$$

where α_A stands for the generators of group $SU(3)$ (see (4.4)), $A = 1 \div 8$. Therefore, intrinsic symmetry transformations can be determined with the help of 36 basic elements

$$L_i = (\sigma_i \otimes I_3) \otimes \gamma_5,\ L_A = (I_2 \otimes \alpha_A) \otimes \gamma_5,$$
$$L_{iA} = (\sigma_i \otimes \alpha_A) \otimes \gamma_5; \qquad (8.4)$$

let us numerate them as follows

$$L_i \to L_1 ... L_3,\ L_A \to L_4 ... L_{11},\ L_{iA} \to L_{12} ... L_{35}. \qquad (8.5)$$

Taking into account the Majorana condition, for 1-parametric transformations of the type $Q_2 = 1 + \Omega L$, we find 21 and 15 restrictions on parameters Ω:

– imaginary

$$\Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_{17},$$
$$\Omega_{18}, \Omega_{19}, \Omega_{20}, \Omega_{21}, \Omega_{22}, \Omega_{23}, \qquad (8.6)$$
$$\Omega_{24}, \Omega_{28}, \Omega_{29}, \Omega_{30}, \Omega_{31}, \Omega_{32}, \Omega_{36};$$

– real

$$\Omega_1, \Omega_9, \Omega_{10}, \Omega_{11}, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{15},$$
$$\Omega_{16}, \Omega_{25}, \Omega_{26}, \Omega_{27}, \Omega_{33}, \Omega_{34}, \Omega_{35}. \qquad (8.7)$$

Only 15 generators referring to real-valued parameters satisfy the Lagrangian condition:

$$L_1 = (\sigma_i \otimes I_3) \otimes \gamma_5,\ L_9 = (I_2 \otimes \alpha_6) \otimes \gamma_5,$$
$$L_{10} = (I_2 \otimes \alpha_7) \otimes \gamma_5,\ L_{11} = (I_2 \otimes \alpha_8) \otimes \gamma_5,$$
$$L_{12} = (\sigma_1 \otimes \alpha_1) \otimes \gamma_5,\ L_{13} = (\sigma_1 \otimes \alpha_2) \otimes \gamma_5,$$
$$L_{14} = (\sigma_1 \otimes \alpha_3) \otimes \gamma_5,\ L_{15} = (\sigma_1 \otimes \alpha_4) \otimes \gamma_5,$$
$$L_{16} = (\sigma_1 \otimes \alpha_5) \otimes \gamma_5,\ L_{25} = (\sigma_2 \otimes \alpha_6) \otimes \gamma_5, \qquad (8.8)$$
$$L_{26} = (\sigma_2 \otimes \alpha_7) \otimes \gamma_5,\ L_{27} = (\sigma_2 \otimes \alpha_8) \otimes \gamma_5,$$
$$L_{33} = (\sigma_3 \otimes \alpha_6) \otimes \gamma_5,\ L_{34} = (\sigma_3 \otimes \alpha_7) \otimes \gamma_5,$$
$$L_{35} = (\sigma_3 \otimes \alpha_8) \otimes \gamma_5;$$

referring to transformations Q_1 15 symmetry generators were given while considering the massive case (4.13).

Thus, for the case of three massless fields, we have found 30-parametric group of intrinsic symmetry. By direct calculation, we can readily find generators all triples with $su(2)$-structure in the set of 30×30. In particular, among 15 generators of the type Q_1 there exist only two such triples

$$Q_1 \quad (J_9, J_{13}, J_{14}), (J_{10}, J_{12}, J_{15}); \qquad (8.9)$$

among 15 generators of the type Q_2 also exist only two such triples:

$$Q_2 \quad (L_9, L_{13}, L_{14}), (L_{10}, L_{12}, L_{15}). \qquad (8.10)$$

9 The System of Four Massless Fields

Let us consider the system of four Dirac fields

$$\begin{gathered} \gamma_\mu \partial_\mu \psi_i = 0 \; (i = 1,2,3,4) \Rightarrow \Gamma_\mu \partial_\mu \Psi = 0, \\ \Psi = (\psi_1^r, \psi_2^r, \psi_3^r, \psi_4^r; \psi_1^i, \psi_2^i, \psi_3^i, \psi_4^i). \end{gathered} \qquad (9.1)$$

Intrinsic symmetry transformations should satisfy relations $[Q_1, \Gamma_\mu]_- = 0$ or $[Q_2, \Gamma_\mu]_+ = 0.$

Because the first condition was studied when considering the massive case, we will examine only the symmetries of type Q_2. Their general structure may be of the form $Q_2 = q \otimes \gamma_5$ where q is an 8×8 complex matrix. Any matrix $q_{8\times 8}$ may be decomposed in the set of 64 elements

$$\begin{gathered} I_8, \gamma_\mu \otimes I_2, \gamma_5 \otimes I_2, \gamma_\mu\gamma_5 \otimes I_2, \gamma_\mu\gamma_\nu \otimes I_2, \\ \gamma_\mu \otimes \sigma_i, \gamma_5 \otimes \sigma_i, \gamma_\mu\gamma_5 \otimes \sigma_i, \gamma_\mu\gamma_\nu \otimes \sigma_i, I_4 \otimes \sigma_i. \end{gathered} \qquad (9.2)$$

The symmetries for this field are determined by 63 generators; they may be listed as follows

$$\begin{gathered} L_\mu \to L_1 ... L_4 \to (\gamma_\mu \otimes I_2) \otimes \gamma_5, \\ L_5 \to L_5 \to (\gamma_5 \otimes I_2) \otimes \gamma_5, \\ L_{\mu 5} \to L_6 ... L_9 \to i(\gamma_\mu\gamma_5 \otimes I_2) \otimes \gamma_5, \\ L_{[\mu\nu]} \to L_{10} ... L_{15} \to i(\gamma_\mu\gamma_\nu \otimes I_2) \otimes \gamma_5, \\ L_{\mu i} \to L_{16} ... L_{27} \to (\gamma_\mu \otimes \sigma_i) \otimes \gamma_5, \\ L_{5i} \to L_{28} ... L_{30} \to (\gamma_5 \otimes \sigma_i) \otimes \gamma_5, \\ L_{[\mu 5 i]} \to L_{31} ... L_{42} \to i(\gamma_\mu\gamma_5 \otimes \sigma_i) \otimes \gamma_5, \\ L_{[\mu\nu] i} \to L_{43} ... L_{60} \to i(\gamma_\mu\gamma_\nu \otimes \sigma_i) \otimes \gamma_5, \\ L_{4i} \to L_{61} ... L_{63} \to (I_4 \otimes \sigma_i) \otimes \gamma_5, \end{gathered} \qquad (9.3)$$

where $i = 1 \div 3, \mu, \nu = 1 \div 4$, $[\mu\nu] = 23, 31, 12, 14, 24, 34.$ All generators have the quadratic minimal equation, $L^2 = I$. The Majorana condition for 1-parametric transformations leads to restrictions on parameters Ω:

28 real parameters

$$\begin{gathered} \Omega_1, \Omega_3, \Omega_7, \Omega_9, \Omega_{11}, \Omega_{14}, \Omega_{16}, \Omega_{18}, \Omega_{20}, \Omega_{22}, \\ \Omega_{24}, \Omega_{26}, \Omega_{29}, \Omega_{32}, \Omega_{34}, \Omega_{36}, \Omega_{38}, \Omega_{40}, \Omega_{42}, \\ \Omega_{44}, \Omega_{46}, \Omega_{48}, \Omega_{50}, \Omega_{53}, \Omega_{55}, \Omega_{57}, \Omega_{59}, \Omega_{62}; \end{gathered} \qquad (9.4)$$

35 imaginary parameters

$$\begin{gathered} \Omega_2, \Omega_4, \Omega_5, \Omega_6, \Omega_8, \Omega_{10}, \Omega_{12}, \Omega_{13}, \Omega_{15}, \Omega_{17}, \Omega_{19}, \\ \Omega_{21}, \Omega_{23}, \Omega_{25}, \Omega_{27}, \Omega_{28}, \Omega_{30}, \Omega_{31}, \Omega_{28}, \Omega_{30}, \\ \Omega_{31}, \Omega_{33}, \Omega_{35}, \Omega_{37}, \Omega_{39}, \Omega_{41}, \Omega_{43}, \Omega_{45}, \Omega_{47}, \\ \Omega_{49}, \Omega_{51}, \Omega_{52}, \Omega_{54}, \Omega_{56}, \Omega_{58}, \Omega_{60}, \Omega_{61}, \Omega_{63}. \end{gathered} \qquad (9.5)$$

The Lagrangian condition (1.6) is satisfied only for 28 one-parametric transformations with real-valued Ω (9.4). Thus, the appropriate symmetries of the type Q_2 are determined by the following 28 generators:

$$\begin{gathered} L_1 = (\gamma_1 \otimes I_2) \otimes \gamma_5, L_3 = (\gamma_3 \otimes I_2) \otimes \gamma_5, \\ L_7 = i(\gamma_2\gamma_5 \otimes I_2) \otimes \gamma_5, L_9 = i(\gamma_4\gamma_5 \otimes I_2) \otimes \gamma_5, \\ L_{11} = i(\gamma_3\gamma_1 \otimes I_2) \otimes \gamma_5, L_{14} = i(\gamma_2\gamma_4 \otimes I_2) \otimes \gamma_5, \\ L_{16} = (\gamma_1 \otimes \sigma_1) \otimes \gamma_5, L_{18} = (\gamma_1 \otimes \sigma_3) \otimes \gamma_5, \\ L_{20} = (\gamma_2 \otimes \sigma_2) \otimes \gamma_5, L_{22} = (\gamma_3 \otimes \sigma_1) \otimes \gamma_5, \\ L_{24} = (\gamma_3 \otimes \sigma_3) \otimes \gamma_5, L_{26} = (\gamma_4 \otimes \sigma_2) \otimes \gamma_5, \\ L_{29} = (\gamma_5 \otimes \sigma_2) \otimes \gamma_5, L_{32} = i(\gamma_1\gamma_5 \otimes \sigma_2) \otimes \gamma_5, \\ L_{34} = i(\gamma_2\gamma_5 \otimes \sigma_1) \otimes \gamma_5, L_{36} = i(\gamma_2\gamma_5 \otimes \sigma_3) \otimes \gamma_5, \\ L_{38} = i(\gamma_3\gamma_5 \otimes \sigma_2) \otimes \gamma_5, L_{40} = i(\gamma_4\gamma_5 \otimes \sigma_1) \otimes \gamma_5, \\ L_{42} = i(\gamma_4\gamma_5 \otimes \sigma_3) \otimes \gamma_5, L_{44} = i(\gamma_2\gamma_3 \otimes \sigma_2) \otimes \gamma_5, \\ L_{46} = i(\gamma_3\gamma_1 \otimes \sigma_1) \otimes \gamma_5, L_{48} = i(\gamma_3\gamma_1 \otimes \sigma_3) \otimes \gamma_5, \\ L_{50} = i(\gamma_1\gamma_2 \otimes \sigma_2) \otimes \gamma_5, L_{53} = i(\gamma_1\gamma_4 \otimes \sigma_2) \otimes \gamma_5, \\ L_{55} = i(\gamma_2\gamma_4 \otimes \sigma_1) \otimes \gamma_5, L_{57} = i(\gamma_2\gamma_4 \otimes \sigma_3) \otimes \gamma_5, \\ L_{59} = i(\gamma_3\gamma_4 \otimes \sigma_2) \otimes \gamma_5, L_{62} = (I_4 \otimes \sigma_2) \otimes \gamma_5. \end{gathered} \qquad (9.6)$$

Their explicit form is omitted because of their bulkiness. All the generators have the dimension 32×32, they may be presented shorter with the use of blocks of dimension 8×8. Collecting together the generators of type Q_2 (9.6) and generators of type Q_1 (5.7), we get the complete symmetry group for the system of 4 massless fields.

The detailed study of the structure of these generators leads to the following conclusions.

1. Among all the generators of types Q_1 and Q_2 (see (5.6) and (9.6)) one can find 56 pairs of triples; in each pair the 6 involved operators obey the commutation rules for algebra $so(4)$. For instance, two examples are

$$(J_1, J_3, J_{11}) \in Q_1, (L_1, L_3, L_{11}) \in Q_2,$$

and

$$(J_7, J_9, J_{14}) \in Q_1, (L_7, L_9, L_{14}) \in Q_2.$$

The complete list of pairs of triples has been found. It should be noted that each triple of the type Q_1 obeys the $su(2)$ algebra.

2. For each 6-element set there exist 10 other sets (each of 6 elements) that commute with the initial set. For instance, the basic set $(J_1, J_3, J_{11}, L_1, L_3, L_{11})$ commutes with the following ones (each with the $so(4)$ structure):

$$\begin{gathered} (J_7, J_9, J_{14}, L_7, L_9, L_{14}), (J_7, J_{40}, J_{55}, L_7, L_{40}, L_{55}), \\ (J_7, J_{42}, J_{57}, L_7, L_{42}, L_{57}), (J_9, J_{34}, J_{55}), L_9, L_{34}, L_{55}), \\ (J_9, J_{36}, J_{57}, L_9, L_{36}, L_{57}), (J_{14}, J_{34}, J_{40}, L_{14}, L_{34}, L_{40}), \\ (J_{14}, J_{36}, J_{42}, L_{14}, L_{36}, L_{42}), (J_{34}, J_{36}, J_{63}, L_{34}, L_{36}, L_{63}), \\ (J_{40}, J_{42}, J_{62}, L_{40}, L_{42}, L_{62}), (J_{35}, J_{57}, J_{62}, L_{35}, L_{57}, L_{62}). \end{gathered}$$

The generators from the basic set do not enter the 10 concomitants sets:

$$[J_{\text{basic}}, J^{A}_{\text{concomit}}]_{-} = 0,\ J_{\text{basic}} \cap J^{A}_{\text{concomit}} = 0,$$

$$[L_{\text{basic}}, L^{A}_{\text{concomit}}]_{-} = 0,\ L_{\text{basic}} \cap L^{A}_{\text{concomit}} = 0,\ A = 1 \div 10.$$

In other words, each basic 6-element set gives rise to the algebra with structure $so(4) \otimes so(4)$.

Conclusion

In the separate paper, we presented the results of the analysis of internal symmetries for quantized Dirac fields, massive and massless ones; also we studied the internal symmetries in presents of electromagnetic and gravitation fields.

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