Проблемы физики, математики и техники, № 3 (52), 2022 **ISSN 2077-8708**

МАТЕМАТИКА

УДК 512.542

DOI: https://doi.org/10.54341/20778708_2022_3_52_82
EDN: GSZHBR

КОНЕЧНЫЕ ГРУППЫ СО СЛАБО СУБНОРМАЛЬНЫМИ ПОДГРУППАМИ ШМИДТА ИЗ НЕКОТОРОЙ МАКСИМАЛЬНОЙ ПОДГРУППЫ

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FINITE GROUPS WITH WEAKLY SUBNORMAL SCHMIDT SUBGROUPS IN SOME MAXIMAL SUBGROUPS

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Аннотация. Подгруппа H называется *слабо субнормальной* в G, если $H = < A, B >$ для некоторой субнормальной в G подгруппы A и полунормальной подгруппы B из G. Здесь подгруппа B называется полунормальной в группе G, если существует подгруппа Y такая, что $G = BY$ и BX – подгруппа для каждой подгруппы X из Y. Конечную ненильпотентную группу, все собственные подгруппы которой нильпотентны, называют группой Шмидта. Если в группе с нильпотентной максимальной подгруппой коммутант силовской 2-подгруппы из максимальной подгруппы содержится в центре силовской 2-подгруппы, то группа будет разрешимой. Если максимальная подгруппа группы ненильпотентна, то в ней существует подгруппа Шмидта. Строение группы, в частности, ее разрешимость, будет зависеть от свойств подгрупп Шмидта из максимальной подгруппы группы. В данной работе устанавливается разрешимость конечной группы, в которой некоторые подгруппы Шмидта из максимальной подгруппы группы слабо субнормальны в группе.

Ключевые слова: *конечная группа*, *разрешимая группа*, *подгруппа Шмидта*, *слабо субнормальная подгруппа*, *максимальная подгруппа*.

Для цитирования: *Зубей*, *Е.В.* Конечные группы со слабо субнормальными подгруппами Шмидта из некоторой максимальной подгруппы / Е.В. Зубей // Проблемы физики, математики и техники. – 2022. – № 3 (52). – С. 82–85. – DOI: https://doi.org/10.54341/20778708_2022_3_52_82. – EDN: GSZHBR

Abstract. A subgroup H is called *weakly subnormal in G* if $H = < A ,B >$ for some subgroup A subnormal in G and seminormal subgroup B of G. Here the subgroup B is called seminormal in G, if there exists a subgroup Y such that $G = BY$ and BX is a subgroup for each subgroup X of Y. Finite non-nilpotent group, whose all proper subgroups are nilpotent are called Schmidt. If in a group with a nilpotent maximal subgroup the derived subgroup of a Sylow 2-subgroup from a maximal subgroup is contained in the center of a Sylow 2-subgroup, then the group is solvable. If the maximal subgroup of a group is non-nilpotent, then in it there is a Schmidt subgroup. The structure of the group itself, in particular, its solvability depends on the properties of Schmidt subgroups from a maximal subgroup of the group. In this paper, we establish the solubility of a finite group under the condition that some Schmidt subgroups from the maximal subgroup groups are weakly subnormal in a group.

Keywords: *finite group*, *solvable group*, *Schmidt subgroup*, *weakly subnormal subgroup*, *maximal subgroup*.

For citation: *Zubei*, *E.V.* Finite groups with weakly subnormal Schmidt subgroups in some maximal subgroups / E.V. Zubei // Problems of Physics, Mathematics and Technics. – 2022. – № 3 (52). – P. 82–85. – DOI: https://doi.org/10.54341/20778708_2022_3_52_82. – EDN: GSZHBR

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [1], [2].

A group G with a nilpotent maximal subgroup M is solvable if the derived subgroup of a Sylow 2-subgroup P of M is contained in the center of P, [2, Theorem IV.7.4]. In particular, a group with nilpotent maximal subgroup of odd order is soluble [3]. These theorems have found a response in many works, see, for example, [4]–[8].

If a maximal subgroup M of a group G is non-nilpotent, then M has a Schmidt subgroup (a non-nilpotent group in which all proper subgroups are nilpotent). The structure of a group G, in particular, the solvability of G, depends on the properties of the Schmidt subgroups of M.

Huang, Hu and Skiba [9] introduced the following definition: a subgroup H is called *weakly subnormal in G* if $H = \langle A, B\rangle$ for some subnormal subgroup A of G and seminormal subgroup B of G. Here the subgroup B is called seminormal in G, if there exists a subgroup Y such that $G = BY$ and BX is a subgroup for each subgroup X of Y. It is clear that if H is seminormal in G, then H is weakly subnormal in G. The converse is false. For example, the alternating group $G = A_4$ has a subgroup of order 2 such that it is subnormal in G, but isn't seminormal in G. Properties of weakly subnormal subgroups are presented in [9], [10]. Groups with seminormal, subnormal, generalized subnormal, or generalized seminormal Schmidt subgroups were considered in [11]–[14]. In this paper, we establish the solubility of a group *G* under the condition that some Schmidt subgroups from a maximal subgroup *M* of *G* are weakly subnormal in *G*.

***Theorem* 0.1.** *Let M be a maximal subgroup of odd index in G and P be a Sylow* 2*-subgroup of M. Suppose that M is* A_4*-free and every Schmidt subgroup of even order of M is weakly subnormal in G. If* $P' \le Z(P)$ *or* $N_G(V) \le M$ *for every* 2*-sub-group V of M which is non-normal in G*, *then G is soluble.*

***Corollary* 0.1** [5, Theorem] *Let P be a Sylow* 2*-subgroup of G. If* $P' \le Z(P)$ *and P is a direct factor of some maximal subgroup of G*, *then G is soluble.*

***Corollary* 0.2** [7, Theorem 1]. *Let M be a maximal subgroup of odd index in G and be* 2*-decomposable. If* $N_G(L) \le M$ *for every* 2*-subgroup L of M which is non-normal in G*, *then G is soluble.*

1 Preliminaries

A Schmidt group is a non-nilpotent group in which every proper subgroup is nilpotent. O.Y. Schmidt [15] initiated the investigations of such groups. He proved that a Schmidt group is biprimary (i. e. its order is divided by only two different primes), one of its Sylow subgroups is normal and the other one is cyclic. Reviews on the structure of the Schmidt groups and their applications in the theory of finite groups are available in [16]. Let us agree to call the $S_{<p,q>}$*-group* [11] a Schmidt group with a normal Sylow *P*-subgroup and a cyclic Sylow *q*-subgroup.

Recall that $A^G = \langle A^g \mid g \in G \rangle$ is the subgroup generated by all subgroups of *G* that are conjugate to *A*.

Denote by $M < \cdot G$, $N \lhd G$, G' and $Z(G)$ a maximal, a normal subgroup, the derived subgroup and the center of *G*, respectively. We denote by $G = [A]B$ a semidirect product of two subgroups *A* and *B* with a normal subgroup *A*. A group *G* is A_4-free if there is no section isomorphic to the alternating group A_4 of degree 4.

***Lemma* 1.1.** (1) *If H is a weakly subnormal* 2*-nilpotent subgroup of G*, *then* H^G *is soluble.*

(2) *Let P be the smallest prime divisor of the order of G. If H is weakly subnormal in G and P does not divide the order of H*, *then P does not divide the order of* H^G.

(3) *If H is a weakly subnormal soluble and* 3 *does not divide the order of H, then* H^G *is soluble.*

Proof. Statements (1)-(2) are true, see [10, Lemma 3.4].

(3) Since *H* is weakly subnormal in *G*, we have $H = \langle A, B \rangle$ for some subnormal subgroup *A* of *G* and seminormal subgroup *B* of *G*. Because

$$H = \langle A, B \rangle \le \langle A^G, B^G \rangle = A^G B^G,$$

it follows that $H^G \le A^G B^G$. Since $A^G \le H^G$ and $B^G \le H^G$, we have $H^G = A^G B^G$. The subgroup B^G is soluble by [11, Lemma 10 (2)]. Since *A* is subnormal in *G*, A^G is soluble. Hence H^G is soluble. □

***Lemma* 1.2** [11, Lemma 1] *If K and D are subgroups of a group G*, *D is normal in K and* K / D *is an* $S_{<p,q>}$*-subgroup*, *then each minimal supplement L to the subgroup D in K has the following properties*:

(1) *L is a P-closed* $\{p,q\}$*-subgroup*;

(2) *all proper normal subgroups of L are nilpotent*;

(3) *L contains an* $S_{<p,q>}$*-subgroup* $[P]Q$ *such that Q is not contained in D and* $L = ([P]Q)^L = Q^L$.

***Lemma* 1.3.** (1) *Every non-P-nilpotent group G contains an* $S_{<p,q>}$*-subgroup for some* $q \in \pi(G)$ [2, IV.5.4].

(2) *Every non-*2*-closed group G contains an* $S_{<q,2>}$*-subgroup for some* $q \in \pi(G)$ [17, p. 34], [18, Corollary 3.1.1].

***Lemma* 1.4.** (1) *If H is a weakly subnormal Schmidt subgroup of G and* H^G *is non-soluble*, *then* $H / Z(H) \simeq A_4$.

(2) *If every* 2*-nilpotent Schmidt subgroup of even order is weakly subnormal in G*, *then G is soluble.*

Proof. (1) If the order of *H* is odd, then H^G has odd order by Lemma 1.1 (2). Then H^G is soluble. If *H* is 2-nilpotent, then H^G is soluble by Lemma 1.1 (2). If 3 does not divide the order of *H*, then H^G is soluble by Lemma 1.1 (3). So, H^G can be non-soluble only if *H* is a 2-closed $\{2,3\}$-subgroup. By properties of Schmidt groups, $H / Z(H) \simeq A_4$ by [16, Theorem 1.2].

(2) If *G* does not have a 2-nilpotent Schmidt subgroups of even order, then *G* is 2-closed by Lemma 1.3 (2); hence *G* is soluble. Let *A* be a 2-nilpotent Schmidt subgroup. Since *A* is weakly subnormal in *G*, we have A^G is soluble by Lemma 1.1 (1).

Let $B = \langle A^G \mid A$ be a 2-nilpotent Schmidt subgroup$\rangle$. Then *B* is soluble and normal in *G*. If G / B hasn't a 2-nilpotent Schmidt subgroup of even order, then G / B is 2-closed by Lemma 1.3 (2) and *G* is soluble.

Let K / B be an $S_{<r,2>}$-subgroup of G / B for some prime r and *L* be a minimal supplement to *B* in *K*. By Lemma 1.2, *L* contains an $S_{<r,2>}$-subgroup *A* such that $A^L = L$. Since

$$K = LB = A^L B \le A^G B \le B,$$

we have a contradiction. □

2 The proof of Theorem 0.1

If M does not have Schmidt subgroups of even order, then M is 2-decomposable by Lemma 1.3. If $P' \le Z(P)$, then by [5, Theorem], G is soluble; if $N_G(V) \le M$ for every 2-subgroup V of M which is non-normal in G, then G is soluble by [7, Theorem 1]. Hence M has Schmidt subgroups of even order.

Let A be a Schmidt subgroup of even order in M. By condition, A is weakly subnormal in G. If A^G is non-soluble, then by Lemma 1.4 (1), $A / Z(A) \simeq A_4$ and M is not A_4-free, a contradiction. Hence A^G is soluble. Since A is arbitrary Schmidt subgroup of even order in M, we have $B = \langle A^G \mid A$ is a Schmidt subgroup of even order in $M\rangle$ is normal in G and is soluble.

Suppose that MB/B is not 2-decomposable. Since $MB/B \simeq M/M\cap B$, it follows that $M/M\cap B$ is not 2-decomposable. By Lemma 1.3, $M/M\cap B$ has a Schmidt subgroup of even order. Let $S/M\cap B$ be an $S_{<r,q>}$-subgroup of even order in $M/M\cap B$ and L be a minimal supplement to $M\cap B$ in S. By Lemma 1.2, L contains an $S_{<r,q>}$-subgroup $[R]Q$ of even order such that Q is not contained in $M\cap B$. Since $[R]Q \le L \le S \le M$, we have $[R]Q \le B$ by the construction of the subgroup B. Hence $[R]Q \le M\cap B$ and $Q \le B$, a contradiction. Therefore, MB/B is 2-decomposable.

Since M is a maximal subgroup of G, it follows that either $MB/B = G/B$, or $B \le M$ and M/B is a maximal subgroup of G/B. If $MB/B = G/B$, then G/B 2-decomposable and since B is soluble, we have G is soluble.

Let $B \le M$ and M/B be maximal subgroups of G/B. Since $|G:M|$ is odd, it follows that $|G/B : M/B|$ is odd.

A subgroup PB/B is a Sylow 2-subgroup of M/B. If $P' \le Z(P)$, then

$$(PB/B)' = P'B/B \le Z(P)B/B \le Z(PB/B).$$

Let $N_G(V) \le M$ for every 2-subgroup V of M which be non-normal in G, and U/B be a 2-subgroup of M/B which is non-normal in G/B. Then $U = U_2B$, where U_2 is a Sylow 2-subgroup of U. We consider $xB \in N_{G/B}(U/B)$. Since $(U/B)^{xB} = U/B$, it follows that $U = U^x$. By Sylow Theorem, $U_2^x = U_2^u$ for $u \in U$. We have that $U^x = U_2^xB = U_2^uB = U$. So,

$$N_{G/B}(U/B) = N_G(U)B/B \le M/B.$$

Therefore, the quotient G/B and its maximal 2-decomposable subgroup M/B satisfy to conditions of the theorem. Then G/B is soluble either by [5, Theorem], or by [7, Theorem 1]. Hence G is soluble. □

The following examples show that we cannot omit the conditions “maximal subgroup is A_4-free” and “$P' \le Z(P)$” in Theorem 0.1.

Example. The group $G = PSL(2,5)$ has Schmidt subgroup S of index 5 such that S is isomorphic to the alternating group A_4 of degree 4. It's obvious that S is maximal and seminormal in G. Besides, S has abelian Sylow 2-subgroup P; hence $P' \le Z(P)$.

The group $G = PSL(2,17)$ has a Sylow 2-subgroup P such that P is isomorphic to the dihedeal group of order 16. It is clear that P is maximal in G and A_4-free. But $P' \not\le Z(P)$.

Remark. Let in the hypothesis of Theorem 0.1 we do not require that the index of the maximal subgroup M be odd and assume that each Schmidt subgroup of M is weakly subnormal in G. Hence keeping the idea of the proof of Theorem 0.1 and applying the assertion [2, Theorem IV.7.4] in the first paragraph we see that the following theorem is true.

***Theorem* 2.1.** *Let M be a maximal subgroup of G and P be a Sylow 2-subgroup of M. Suppose that $P' \le Z(P)$ and M is A_4-free. If every Schmidt subgroup of M is weakly subnormal in G, then G is soluble.*

***Corollary* 2.1** [2, Theorem IV.7.4]. *Let M be a maximal subgroup of G and P be a Sylow 2-subgroup of M. If M is nilpotent and $P' \le Z(P)$, then G is soluble.*

***Corollary* 2.2.** *Every group with nilpotent maximal subgroup of odd order is soluble.*

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Поступила в редакцию 06.08.2022.

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