МАТЕМАТИКА

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ГРАНИЦЫ ВАЛИДНОСТИ АСИМПТОТИЧЕСКИХ АППРОКСИМАЦИЙ ДЛЯ РАСЩЕПЛЯЮЩЕГО ПРЕОБРАЗОВАНИЯ ТРЕХТЕМПОВЫХ ЛИНЕЙНЫХ СТАЦИОНАРНЫХ СИНГУЛЯРНО ВОЗМУЩЕННЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

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ASYMPTOTIC APPROXIMATIONS VALIDITY BOUNDARIES FOR DECOUPLING TRANSFORMATION OF THREE-TIME-SCALE LINEAR TIME-INVARIANT SINGULARLY PERTURBED SYSTEMS WITH DELAY

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Аннотация. Для трехтемповых линейных стационарных сингулярно возмущенных систем с запаздыванием по состоянию развивается метод разделения движений на основе невырожденного преобразования типа Чанг. Введены асимптотические приближения для полностью разделенных подсистем рассматриваемой системы с трехвременными масштабами, построены и доказаны границы значений малых параметров сингулярности, гарантирующие справедливость асимптотических представлений решений матричных операторных уравнений, лежащих в основе преобразования, асимптотических приближений для расщепляющего преобразования и матричных операторов расщепленной системы. Приведен иллюстративный пример.

Ключевые слова: *сингулярно возмущенная система*, *трехтемповая система*, *запаздывание*, *расщепляющее преобразование*, *декомпозиция*, *асимптотическое приближение*, *оценка параметра.*

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Abstract. For time-invariant singularly perturbed control systems with state delay the method of separation of movements is evolved on the basis of Chang-type non-degenerate transformation. Asymptotic approximations for completely separated subsystems of the considered singularly perturbed system with three-time scales are introduced, boundaries of values of small singularity parameters are constructed and proved, which guarantee the validity of asymptotic representations and estimates of solutions underlying matrix operator equations, asymptotic approximations for the decoupling transformation and matrix operators of the split system. An illustrative example is given.

Keywords: *singularly perturbed system*, *three-time-scale system*, *time delay*, *decoupling transformation*, *decomposition*, *asymptotic approximation*, *parameter estimate.*

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Introduction

Singularly Perturbed Systems (SPS) are a common occurrence specially in contemporary application in engineering, quantum mechanics, optimal control, etc. [1]–[3]. High dimensionality and rigidity of such systems due to the precedence of small parameters which can be sourced by a wide range of physical parameters such as friction factors, viscosity and other system parameters, have made analysis of solving such systems harder even in modern mathematics.

Time scale separation of the Singularly Perturbed Systems is a widely studied area due to the possibility of reducing the dimension of the simulated systems for the purpose of analysing and synthesizing systems. Time scale separation of a linear singularly-perturbed continuous-time varying systems without delay has been introduced with Chang's transformation [4]–[6] with a nondegenerate change of variables. Generalization of the Change-type transformation on a two-time scale singular perturbed time-invariant system without delay has been done in [7]–[9] with two step change of variables. There have been several studies in the area on the generalization of Chang's transformation to two, three and multi-time scale Singularly Perturbed systems in slitting the subsystems depending on the space of the variables which leads to lower

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dimensional systems which are lesser in complexity in solving. Further, studies related to Singular Perturbed Systems with delays have been conducted in [10], [11] where Chang type transformation has been generalized to obtained the subsystems with delays.

Methods of solving Singularly Perturbed Systems is one of the widely researched areas in mathematics and many approaches of solving Singularly Perturbed Systems have been introduced by mathematicians. Majorly those approaches are based either on classical numerical approaches or on asymptotic methods, and hybrid approaches have also been considered. As far as the numerical methods of solving Singularly Perturbed Systems are concerned, only the quantitative information of considered problems can be obtained, while some explicit information of the quantitative behaviour of the family of the problem can be acquired with the application of asymptotic methods [12]. In complex SPSs, it is difficult to develop efficient numerical methods for SPS due to the rigidity of such systems. When the asymptotic methods are considered, the main term of the asymptotic approximations contains necessarily the essentials for the indication of the qualitative behaviour of the solution and in some cases is capable of replacing the exact solution to the problem. Thus, the development of asymptotic methods contributes to the development of accurate numerical methods, as the knowledge of the structure of the solution helps in the development of numerical methods for solving complex problems. The simulation results of the study [13] indicate that the asymptotic method is effective in the approximation of sub-systems of SPS without compromising the qualitative behaviours of the solutions. So, it is important to construct a splitting transformation and its asymptotic approximation as a method for decomposing complex systems, a way to eliminate the rigidity of such systems when developing effective numerical methods.

The study [10] generalized the transformation introduced by Chang in [4], to a linear time invariant system with delay in the slow state variables where the transformation has been constructed in the form of an asymptotic series. In [11] a two-time scale SPS with multiple commensurate delays in the slow state variables is considered and Chang's transformation has been applied. An SPS is decomposed into its fast and slow subsystems and matrix-valued operators are approximated with respect to the small parameter in an asymptotic series form. In a later study [12] for a similar SPS, schema of the Chang's transformation of [4] is applied and existence/continuity of asymptotic approximations to the solutions of matrix-valued operator of the transformation is proved and discussed.

In a similar study [14], [15] decomposition and asymptotic approximation of the three-time-scale singularly perturbed systems with multiple commensurate delays in the slow state variables has been discussed. But there are few studies carried out considering the measure of reliability of the asymptotic approximations for the SPS and, specially no studies have been conducted for three-time-scale SPSs with delay.

However, extension of the splitting transformation proposed by Chang to systems with delay is generally impossible. This is related to the existence and properties of fast and slow manifolds for such systems [15]. Extending the splitting transformation to systems with delay in the case when this is possible requires, in particular, substantiating the asymptotic representation of solutions of matrix operator equations that depend on small parameters and obtaining corresponding estimates for small parameters. For three-scale systems with delay, this problem is solved for the first time.

So, this work extends the results in [14], [15] by deriving of a measure of the reliability (validity) of the asymptotic approximation of the decoupled due to the splitting transformation of the considered TSPLTISD.

The study [14] focuses on the construction and substantiation of a nondegenerate transformation, which splits a three time-scale linear time-invariant singularly perturbed system with perturbation parameters of two different orders of magnitude (small parameters of two different orders at the highest derivatives of a part of the variables) and with a delay in slow state variables into three regularly dependent on small parameters independent subsystems of smaller dimensions than the original ones: relatively fast, fastest and slow variables and the matrix operator equations are obtained, which must be satisfied by the elements of a non-degenerate transformation, so that as a result of its application, unrelated subsystems of different rates are obtained.

As the continuation of the study [14], in [15], it is discussed that the decoupling transformation formed in the study [14] can be constructed with any degree of accuracy in the form of an asymptotic expansion in powers of small parameters. For this, the proof of the solvability of the previously obtained matrix operator equations is discussed. Further it is discussed that decomposed subsystems can be represented in the form of asymptotic expansions in powers of small parameters, and iterative schemes can be determined for finding the terms of the asymptotic series. Extensive elaboration of the proof of the theorems related to [15] shall be published in the forthcoming publications of the authors.

The interest of this study is the construction of the boundaries with a small parameter with respect to the reliability / validity of asymptotic approximations of three time scale singularly perturbed system with multiple commensurate delays in the slow state variables. Generalization of Chang's type transformation for a Three Time Scale Singularly Perturbed System with multiple commensurate delays in the slow state variable has been discussed by the previous research works by the authors and the construction of iterative schemas for the asymptotic approximation for the decoupled sub-systems depending on the tempo of the variables has also been carried out in the previous works [14], [15].

In this paper, with the use of a contraction mapping principle and the fixed point theorem the upper bounds on small singularity parameters guarantying the validity of asymptotic approxi-mations for the decoupling transformation and matrix operators of the decoupled system are proven.

1 Statement of the problem

A three-time-scale singularly perturbed linear time-invariant control system with multiple commensurate delays in the slow state variables (TSPLTISD) is considered as defined here:

$$\begin{aligned} \dot{x}(t) &= \sum_{j=0}^{l} A_{11j} x(t-jh) + A_{12} y(t) + \\ &+ A_{13} z(t) + B_1 u(t),\ x \in R^{n_1},\ u \in R^r, \\ \varepsilon_1 \dot{y}(t) &= \sum_{j=0}^{l} A_{21j} x(t-jh) + A_{22} y(t) + \\ &+ A_{23} z(t) + B_2 u(t),\ y \in R^{n_2}, \\ \varepsilon_2 \dot{z}(t) &= \sum_{j=0}^{l} A_{31j} x(t-jh) + A_{32} y(t) + \\ &+ A_{33} z(t) + B_3 u(t),\ z \in R^{n_3},\ t \ge 0, \end{aligned} \tag{1.1}$$

with initial conditions

$$\begin{gathered} x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0, \\ x(\theta) = \varphi(\theta),\ \theta \in [-h, 0). \end{gathered} \tag{1.2}$$

Here $A_{i1j}, A_{i2}, A_{i3}, B_i,\ i = \overline{1,3},\ j = \overline{0,l}$ are constant matrices with appropriate dimensions, $h = const > 0$ – delay, $0 < \varepsilon_2 \ll \varepsilon_1 \ll 1$ – small parameters, that describe the time-scale separation, $\varphi(\theta),\ \theta \in [-h, 0)$ – piecewise continuous n_1-vector function, $u(t)$ – piecewise continuous on *T*, *r*-vector control function, $x_0 \in R^{n_1}$, $y_0 \in R^{n_2}$.

Note that since $\varepsilon_2 \ll \varepsilon_1 \ll 1$, then $\varepsilon_1 \varepsilon_2 \ll \varepsilon_2$, $\frac{\varepsilon_2}{\varepsilon_1} \ll 1$, $\frac{\varepsilon_2}{\varepsilon_1} \xrightarrow[\varepsilon_1 \to 0]{} 0.$

The presence of two small parameters of different orders of smallness in the form of factors at the derivatives of some of the variables determines the different-rate nature of the change in the phase coordinates in the vicinity of the point. So, *x* is slow, *y* is fast and *z* is the fastest variables.

Let $p \triangleq \frac{d}{dt}$ – differentiation operator, e^{-ph} – delay operator:

$$e^{-ph} v(t) = v(t-h),$$
$$e^{-jph} v(t) = v(t-jh).$$

Introducing matrix operators

$$A_{i1}\left(e^{-ph}\right) \triangleq \sum_{j=0}^{l} A_{i1j} e^{-jph},\ i = \overline{1,3},$$

block matrix operator and block matrix

$$A\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) = \begin{bmatrix} A_{11}\left(e^{-ph}\right) & A_{12} & A_{13} \\ \dfrac{A_{21}\left(e^{-ph}\right)}{\varepsilon_1} & \dfrac{A_{22}}{\varepsilon_1} & \dfrac{A_{23}}{\varepsilon_1} \\ \dfrac{A_{31}\left(e^{-ph}\right)}{\varepsilon_2} & \dfrac{A_{32}}{\varepsilon_2} & \dfrac{A_{33}}{\varepsilon_2} \end{bmatrix},$$

$$B(\varepsilon_1, \varepsilon_2) = \begin{bmatrix} B_1 \\ \dfrac{B_2}{\varepsilon_1} \\ \dfrac{B_3}{\varepsilon_2} \end{bmatrix}. \tag{1.3}$$

For $\varepsilon_2 > 0$ rewrite the system (1) in the equivalent matrix-operator form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = A\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + B(\varepsilon_1, \varepsilon_2) u(t).$$

For simplicity further, where this does not lead to an ambiguous understanding, in the sequel the arguments for matrix functions $A_{i1}\left(e^{-ph}\right),\ i = \overline{1,3}$, etc. will be omitted. We will use $\|A\|_\infty \triangleq \max_i \sum_j |a_{ij}|$.

In [14] for the system (1.1) a decoupling transformation that decomposes TSPLTISD (1.1) to three independent subsystems of lower dimension (relatively faster, fast and slow variables) has been constructed.

For the decoupling of the TSPLTISD (1.1) next transformation (change of variables) has been introduced in [14]

$$\begin{bmatrix} \xi(t) \\ \eta(t) \\ \beta(t) \end{bmatrix} = T\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix},$$

$$\xi(t) \in R^{n_1},\ \eta(t) \in R^{n_2},\ \beta(t) \in R^{n_3},\ t \in T \tag{1.4}$$

where

$$T\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) = \tag{1.5}$$

$$= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \varepsilon_1 \varepsilon_2 H_1 H_3 - \varepsilon_2 H_2 \\ L_1 - \varepsilon_2 H_3 L_2 & I_n - \varepsilon_2 H_3 L_3 & -\varepsilon_2 H_3 \\ L_2 & L_3 & I_p \end{bmatrix},$$

$$\bar{A}_{11}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) = I_m - \varepsilon_1 H_1 \left(L_1 - \varepsilon_2 H_3 L_2\right) - \varepsilon_2 H_2 L_2,$$

$$\bar{A}_{12}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) = \varepsilon_1 \varepsilon_2 H_1 H_3 L_3 - \varepsilon_2 H_2 L_3 - \varepsilon_1 H_1,$$

$$L_i\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right),\ H_i\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right),\ i = \overline{1,3}$$

are matrix operators depending on the parameters

$\varepsilon_1, \varepsilon_2 > 0$ and satisfying the equations (with $\mu = \varepsilon_2 / \varepsilon_1$):

$$\mu L \begin{bmatrix} \varepsilon_1 A_{11} & \varepsilon_1 A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \mu L \begin{bmatrix} \varepsilon_1 A_{13} \\ A_{23} \end{bmatrix} L + \\ + (A_{31}, A_{32}) - A_{33} L = 0,$$

$$L(\varepsilon_1, \mu, e^{-ph}) = \left[L_2(\varepsilon_1, \mu, e^{-ph}), L_3(\varepsilon_1, \mu, e^{-ph}) \right], \quad (1.6)$$

$$A_{31} - A_{33} L_2 + \varepsilon_2 L_2 (A_{11} - A_{13} L_2) + \\ + \mu L_3 (A_{21} - A_{23} L_2) = 0,\ L_2(\varepsilon_1, \mu, e^{-ph}) \in \mathbb{C}^{n_3 \times n_1}, \quad (1.7)$$

$$A_{32} - A_{33} L_3 + \varepsilon_2 L_2 (A_{12} - A_{13} L_3) + \\ + \mu L_3 (A_{22} - A_{23} L_3) = 0,\ L_3(\varepsilon_2, \mu, e^{-ph}) \in \mathbb{C}^{n_3 \times n_2}, \quad (1.8)$$

$$A_{21} - (A_{22} - A_{23} L_3) L_1 - A_{23} L_2 + \\ + \varepsilon_1 L_1 (A_{11} - A_{13} L_2 - (A_{12} - A_{13} L_3) L_1) = 0, \\ L_1(\varepsilon_1, \mu, e^{-ph}) \in \mathbb{C}^{n_2 \times n_1}, \quad (1.9)$$

$$A_{23} - H_3 A_{33} + \\ + \varepsilon_2 (L_1 A_{12} H_3 - H_3 L_2 A_{13} - L_1 A_{13} L_3 H_3) + \quad (1.10) \\ + \varepsilon_1 L_1 A_{13} + \mu (A_{22} H_3 - A_{23} L_3 H_3 - H_3 L_3 A_{23}) = 0,$$

$$A_{12} - H_1 (A_{22} - A_{23} L_3) - A_{13} L_3 + \\ + \varepsilon_1 (A_{11} - A_{13} L_2 - (A_{12} - A_{13} L_3) L_1) H_1 - \quad (1.11) \\ - \varepsilon_1 H_1 L_1 (A_{12} - A_{13} L_3) = 0,$$

$$A_{13} + \varepsilon_2 H_2 (A_{11} - A_{13} L_2 - A_{12} L_1 + A_{13} L_3 L_1) + \\ + \mu A_{12} H_3 - \mu A_{13} L_3 H_3 - \quad (1.12) \\ - H_2 (A_{33} + \varepsilon_2 L_2 A_{13} + \mu L_3 A_{23}) = 0, \\ H_3(\varepsilon_1, \mu, e^{-ph}) \in \mathbb{C}^{n_2 \times n_3}, \\ H_1(\varepsilon_1, \mu, e^{-ph}) \in \mathbb{C}^{n_1 \times n_2}, \\ H_2(\varepsilon_1, \mu, e^{-ph}) \in \mathbb{C}^{n_2 \times n_3}.$$

The matrix operators

$$L_i(\varepsilon_1, \varepsilon_2, e^{-ph}),\ H_i(\varepsilon_1, \varepsilon_2, e^{-ph}),\ i = \overline{1,3}$$

used in this transformation satisfy the algebraic equations of Riccati and Sylvester.

This article, focuses on deriving the boundaries of the values of small parameters, for which approximations of the (1.6)–(1.12) solutions are valid.

2 Measure of the Reliability of the Asymptotic Approximations

To prove the boundaries of the values of small parameters let us note,

$$A^{12}(\varepsilon_1, e^{-ph}) \triangleq \begin{pmatrix} \varepsilon_1 A_{11}(e^{-ph}) & \varepsilon_1 A_{12} \\ A_{21}(e^{-ph}) & A_{22} \end{pmatrix},$$

$$A^{13}(\varepsilon_1, e^{-ph}) \triangleq \begin{pmatrix} \varepsilon_1 A_{13}(e^{-ph}) \\ A_{23}(e^{-ph}) \end{pmatrix}$$

and rewrite (1.6) as

$$\mu L A^{12}(\varepsilon_1, e^{-ph}) - \\ - \mu L A^{13}(\varepsilon_1, e^{-ph}) L + (A_{31}, A_{32}) = A_{33} L. \quad (2.1)$$

***Lemma* 2.1.** *Suppose that* $det\, A_{33} \neq 0$,

$$det\, [A_{22} - A_{23} L_3^{00}] \neq 0.$$

Then for all $\mu \in [0, \mu^*)$ *such that*

$$\mu^* = \frac{1}{\left\| A_{33}^{-1} \right\| \left(a + bd + 2(abd)^{1/2} \right)}, \quad (2.2)$$

where $a \triangleq \left\| A^{12}(1, e^{-ph}) - A^{13} L^{00}(e^{-ph}) \right\|$, $b \triangleq \left\| A^{13}(1) \right\|$,

$$d \triangleq \left\| L^{00}(e^{-ph}) \right\|,\ b(\varepsilon_1) \triangleq \left\| A^{13}(\varepsilon_1) \right\|,$$

$$a(\varepsilon_1) \triangleq \left\| A^{12}(\varepsilon_1, e^{-ph}) - A^{13}(\varepsilon_1, e^{-ph}) L^0(e^{-ph}) \right\|,$$

there are unique continuous functions depending on μ, ε_1,

$$L(\varepsilon_1, \mu, e^{-ph}) = \left[L_2(\varepsilon_1, \mu, e^{-ph}), L_3(\varepsilon_1, \mu, e^{-ph}) \right],$$

satisfying the equations (1.6)–(1.8), *that could be represented in asymptotic series form*:

$$L(\varepsilon_1, \mu, e^{-ph}) = \sum_{m=0}^{M} \mu^m L^m(\varepsilon_1, e^{-ph}) + O(\mu^{M+1}), \quad (2.3)$$

$$L_i(\varepsilon_1, \mu, e^{-ph}) = \sum_{m=0}^{M} \mu^m L_i^m(\varepsilon_1, e^{-ph}) + O(\mu^{M+1}), \\ L_i^m(\varepsilon_1, e^{-ph}) = \sum_{n=0}^{m} \varepsilon_1^n L_i^{nm}(e^{-ph}),\ i = 2,3 \quad (2.4)$$

where the terms $L^m(\varepsilon_1, e^{-ph})$, *according to iterative schemes*

$$L^m(\varepsilon_1, e^{-ph}) = \sum_{n=0}^{m} \varepsilon_1^n L^{nm}(e^{-ph}), \quad (2.5)$$

where $L^{00} = L^0 = A_{33}^{-1} [A_{31}, A_{32}]$,

$$L^{nm} = A_{33}^{-1} \Bigg[L^{n-1,m-1} A_{1,12} + L^{n,m-1} A_{2,12} - \\ - \sum_{j=0}^{m-1} \sum_{i=n-m+j}^{n-1} L^{ij} A_{1,3} L^{n-i-1,m-j-1} - \quad (2.6) \\ - \sum_{j=0}^{m-1} \sum_{i=n-m+j+1}^{n} L^{ij} A_{2,3} L^{n-i,m-j-1} \Bigg],\ m \geq 1,\ n \leq m,$$

$$A_{1,12}(e^{-ph}) = \begin{bmatrix} A_{11}(e^{-ph}) & A_{12} \\ 0 & 0 \end{bmatrix},$$

$$A_{2,12} = \begin{bmatrix} 0 & 0 \\ A_{21}(e^{-ph}) & A_{22} \end{bmatrix},\ A_{1,3} = \begin{bmatrix} A_{13} \\ 0 \end{bmatrix},\ A_{2,3} = \begin{bmatrix} 0 \\ A_{23} \end{bmatrix}.$$

Proof. The existence and uniqueness of the continuous functions

$$L(\varepsilon_1, \mu, e^{-ph}) = \left[L_2(\varepsilon_1, \mu, e^{-ph}), L_3(\varepsilon_1, \mu, e^{-ph}) \right]$$

in the form of (2.3), (2.4) is similar to that is discussed in [9].

The iterative schemes (2.4), (2.6) and representation (2.5) can be proven by substituting (2.3) and (2.5) into the equations (2.1) and comparing the coefficients of equal powers of μ of the resulting equations.

Let us prove the assessment (2.2).

Denoting

$$L^0\left(\varepsilon_1, e^{-ph}\right) \triangleq L\left(\varepsilon_1, 0, e^{-ph}\right) = = L^{00}\left(\varepsilon_1, e^{-ph}\right) = A_{33}^{-1}\left[A_{31}, A_{32}\right],$$

introducing

$$D\left(\varepsilon_1, \mu, e^{-ph}\right) = L\left(\varepsilon_1, \mu, e^{-ph}\right) - L^0\left(\varepsilon_1, e^{-ph}\right)$$

and taking into account $A_{33}L^0\left(\varepsilon_1, e^{-ph}\right) = \left[A_{31}, A_{32}\right]$, rewriting (2.1) as

$$D = \mu A_{33}^{-1}\left[L^0\left(A^{12} - A^{13}L^0\right) + D\left(A^{12} - A^{13}L^0\right) - -L^0 A^{13} D - D A^{13} D\right] \triangleq f\left(D\left(\varepsilon_1, \mu, e^{-ph}\right)\right). \quad (2.7)$$

Further similar as in [9] consider

$$\Gamma = \left\{ D\left(\varepsilon_1, \mu, e^{-ph}\right) : \|D\| \le \left(\frac{a(1)d}{b(1)}\right)^{1/2} \right\},$$

and taking into account the properties of the norm, it can be seen that,

$$a(\varepsilon_1) < a(1) = a, \ \ b(\varepsilon_1) < b(1) = b. \quad (2.8)$$

Then taking into account (2.4), for $\varepsilon_1 \ll 1$ it can be shown that if the condition,

$$\left\|\mu A_{33}^{-1}\right\| \le \frac{1}{a(1) + b(1)d + 2\left(a(1)b(1)d\right)^{1/2}}$$

is satisfied then, $\left\|f\left(D(\varepsilon_1, \mu, e^{-ph})\right)\right\| \le \left(\frac{a(1)d}{b(1)}\right)^{1/2}$ and for any $D, \tilde{D} \in \Gamma$ under the condition

$$\left\|f(D) - f(\tilde{D})\right\| \le \left\|D - \tilde{D}\right\|,$$

i. e. $f(D)$ is contraction mapping.

By the fixed point theorem, the equation (2.7) has a unique solution in Γ, that can be a successive approximation starting with any $D^0 \le \left(\frac{ad}{b}\right)^{1/2}$ if,

$$\mu < \frac{1}{\left\|A_{33}^{-1}\right\|\left(a + bd + 2\left(abd\right)^{1/2}\right)}. \qquad \square$$

***Lemma* 2.2.** *Suppose that* $\det A_{33} \ne 0$, *det* $[A_{22} - A_{23}L_3^{00}] \ne 0$, $\mu < \varepsilon_1$, $\mu \in \left[0, \mu^*\right)$, μ^* *satisfies* (2.1). *Then for all* $\varepsilon_1 \in \left[0, \varepsilon_1^*\right)$ *such that*

$$\varepsilon_1^* = \frac{1}{\left\|[A_{22} - A_{23}L_3]^{-1}\right\|\left(a + bl + 2\sqrt{(k + al)b}\right)}, \quad (2.9)$$

where $l \triangleq \left\|L_1^0\right\|$ *and* a, b, k *are any non-negative numbers satisfying* $b \ge \tilde{b} \triangleq \left\|A_{13}L_3 - A_{12}\right\|$,

$$a \ge \tilde{a} \triangleq \left\|A_{11} + A_{13}L_3L_1^0 - A_{13}L_2 - A_{12}L_1^0\right\|,$$

$k \ge \tilde{k} \triangleq \left\|A_{23}\frac{1}{\mu}\left(D_2 - D_3L_1^{00}\right)\right\|$, *there are unique continuous functions depending on* μ, ε_1, $L_1\left(\varepsilon_1, \mu, e^{-ph}\right)$, *satisfying the equation* (1.8), *that could be represented in an asymptotic series form*:

$$L_1\left(\varepsilon_1, \mu, e^{-ph}\right) = \sum_{n=0}^{N} \varepsilon_1^n L_1^n\left(\mu, e^{-ph}\right) + O\left(\varepsilon_1^{N+1}\right),$$
$$L_1^n\left(\mu, e^{-ph}\right) = \sum_{m=0}^{\infty} \mu^m L_1^{nm}\left(e^{-ph}\right), \quad (2.10)$$

where the terms, $L_1^{nm}\left(e^{-ph}\right)$ *of the asymptotic series can be found according to iterative schemes* (2.11).

$$L_1^{mn}\left(e^{-ph}\right) = \left(A_{22} - A_{23}L_3^{00}\right)^{-1} \times \times\Bigg(L_1^{n-1,m}A_{11} - \sum_{j=1}^{n}\sum_{k=0}^{m} L_1^{j-1,k} \times \times\left(A_{12}L_1^{n-j,m-k} - A_{13}\sum_{r=0}^{n}\sum_{s=r}^{m} L_3^{r-j,s-k}L_1^{n-r-j,m-s-k}\right) + \quad (2.11)$$
$$+A_{23}\sum_{j=0}^{n}\sum_{k=j}^{m} L_3^{jk}L_1^{n-j,m-k} - A_{23}L_2^{mn} + +\sum_{j=1}^{n}\sum_{k=0}^{m-n} L_1^{j-1,k}A_{13}L_2^{n-j,m-k}\Bigg), m > 0, \ n > 0;$$

with initial conditions

$$L_1^{00}(e^{-ph}) = \left(A_{22} - A_{23}A_{33}^{-1}A_{32}\right)^{-1}\left(A_{21} - A_{23}A_{33}^{-1}A_{31}\right),$$
$$L_1^{mn}\left(e^{-ph}\right) = 0, \ n < 0 \vee m < 0. \quad (2.12)$$

Proof. The existence and uniqueness of the continuous functions $L_1\left(\varepsilon_1, \mu, e^{-ph}\right)$ in the form of (2.10), (2.11) are similar to that is discussed in [9].

The iterative schemes (2.10), (2.11) and the representation in (2.10) can be proven by substituting (2.10) into the equations (1.9) and comparing the coefficients of equal powers of μ, ε_1 of the resulting equations.

Let us prove the assessment (2.9).

Introducing

$$D_i\left(\varepsilon_1, \mu, e^{-ph}\right) = L_i\left(\varepsilon_1, \mu, e^{-ph}\right) - L_i^{00}\left(e^{-ph}\right), \ i = 1, 2, 3$$

and taking into account

$$L_1^{00}(e^{-ph}) = \left[A_{22} - A_{23}L_3^{00}\right]^{-1}\left[A_{21} - A_{23}L_2^{00}\right],$$

rewrite (1.9) as

$$D_1 = \varepsilon_1\left[A_{22} - A_{23}L_3\right]^{-1}\Bigg[-\frac{\mu}{\varepsilon_1}A_{23}\left(\bar{D}_2 - \bar{D}_3L_1^{00}\right) + +L_1^{00}\left(A_{11} - A_{13}L_2\right) - L_1^{00}\left(A_{12} - A_{13}L_3\right)L_1^{00} - -L_1^{00}\left(A_{12} - A_{13}L_3\right)D_1 + +D_1\left(\left(A_{11} - A_{13}L_2\right) - \left(A_{12} - A_{13}L_3\right)L_1^{00}\right) - -D_1\left(A_{12} - A_{13}L_3\right)D_1\Bigg] = \quad (2.13)$$
$$\triangleq f_1\left(D_1\left(\varepsilon_1, \mu, e^{-ph}\right)\right), \ \bar{D}_i = \frac{1}{\mu}D_i, \ i = 2, 3.$$

Note that due to (2.7) for $\mu < \varepsilon_1$, $\bar{D}_i$, $i = 2, 3$, are bounded as $\mu \to 0$.

Let $\Gamma_1 = \left\{ D_1\left(\varepsilon_1, \mu, e^{-ph}\right) : \|D_1\| \le \left(\frac{k+al}{b}\right)^{1/2} \right\}$, then for $D_1 \in \Gamma_1$ from (2.13) under condition $\frac{\mu}{\varepsilon_1} < 1$ we can derive

$$\left\| f_1\left(D_1\right)\right\| \le \varepsilon_1 \left\| \left[A_{22} - A_{23}L_3\right]^{-1} \right\| \times$$
$$\times \left[a + bl + 2\sqrt{(k+al)b}\right]\left(\frac{k+al}{b}\right)^{1/2}.$$

So, if the condition

$$\varepsilon_1 \le \frac{1}{\left\| \left[A_{22} - A_{23}L_3\right]^{-1} \right\| \left\| \left[a + bl + 2\sqrt{(k+al)b}\right] \right\|} \qquad (2.14)$$

is satisfied and $D_1 \in \Gamma_1$ then $\left\| f_1\left(D_1\right)\right\| \le \left(\frac{k+al}{b}\right)^{1/2}$ i. e. $f\left(D_1\right) \in \Gamma_1$.

Let $D_1, \tilde{D}_1 \in \Gamma_1$. Under the condition (2.9):

$$\left\| f_1(D_1) - f_1(\tilde{D}_1) \right\| = \varepsilon_1 \left\| \left[A_{22} - A_{23}L_3\right]^{-1} \right\| \times$$
$$\times \Big\| \left(D_1 - \tilde{D}_1\right)\left(\left(A_{11} - A_{13}L_2\right) - \left(A_{12} - A_{13}L_3\right)L_1^{00}\right) -$$
$$- L_1^{00}\left(A_{12} - A_{13}L_3\right)\left(D_1 - \tilde{D}_1\right) -$$
$$- \left(D_1 - \tilde{D}_1\right)\left(A_{12} - A_{13}L_3\right)D_1 +$$
$$+ \tilde{D}_1\left(A_{12} - A_{13}L_3\right)\left(D_1 - \tilde{D}_1\right) \Big\| \le$$
$$\le \varepsilon_1 \left\| \left[A_{22} - A_{23}L_3\right]^{-1} \right\| \left\| D_1 - \tilde{D}_1 \right\| \times$$
$$\times \Big\| \left(A_{11} - A_{13}L_2\right) - \left(A_{12} - A_{13}L_3\right)L_1^{00} - L_1^{00}\left(A_{12} - A_{13}L_3\right) -$$
$$- \left(A_{12} - A_{13}L_3\right)D_1 + \tilde{D}_1\left(A_{12} - A_{13}L_3\right) \Big\| \le$$
$$\le \varepsilon_1 \left\| \left[A_{22} - A_{23}L_3\right]^{-1} \right\| \left\| D_1 - \tilde{D}_1 \right\| \times$$
$$\times \left(a + lb + 2\sqrt{(k+al)b}\right) \le \left\| D_1 - \tilde{D}_1 \right\|,$$

i. e., $f_1(D_1)$ is a contraction mapping.

By the fixed point theorem, the equation (2.13) has a unique solution in Γ_1, that can be a successive approximation starting with any $D_1^0 \le \left(\frac{k+al}{b}\right)^{1/2}$ if

$$\varepsilon_1 \le \frac{1}{\left\| \left[A_{22} - A_{23}L_3\right]^{-1} \right\| \left\| \left[a + bl + 2\sqrt{(k+al)b}\right] \right\|}. \qquad \square$$

Let matrix operators $H_3^{nm}\left(e^{-ph}\right)$, $H_2^{nm}\left(e^{-ph}\right)$, $H_1^{nm}\left(e^{-ph}\right)$ be found according to the following iterative scheme:

$$H_3^{mn}\left(e^{-ph}\right) = \left(\sum_{j=1}^{n}\sum_{k=1}^{m} L_1^{j-1,k-1}\left(A_{12}H_3^{n-j,m-k} - \right.\right.$$
$$\left. - A_{13}\sum_{r=0}^{n}\sum_{s=r}^{m} L_3^{r-j,s-k} H_3^{n-r-j,m-s-k} \right) -$$
$$- A_{23}\sum_{j=0}^{n}\sum_{k=j}^{m} L_3^{j,k-1} H_3^{n-j,m-k} + A_{22}H_3^{n,m-1} -$$
$$- \sum_{j=1}^{n}\sum_{k=1}^{m-n} H_3^{j-1,k-1} L_2^{n-j,m-k} A_{13} + \qquad (2.15)$$
$$\left. + \sum_{j=1}^{n}\sum_{k=0}^{m-n} H_3^{j,k-1} L_3^{n-j,m-k} A_{23} + L_1^{n-1,m} A_{13} \right) A_{33}^{-1},$$
$$m > 0,\ n > 0,$$

with initial conditions

$$H_3^{00}(e^{-ph}) = A_{22}A_{33}^{-1},$$
$$H_3^{0,m}(e^{-ph}) =$$
$$= \left[A_{22}H_3^{0,m-1} - \sum_{k=1}^{m} L_3^{0,k-1} H_3^{0,m-k} + \right.$$
$$\left. + A_{23}\sum_{k=1}^{m-n} L_3^{0,k-1} H_3^{0,m-k} \right] A_{33}^{-1},$$
$$H_3^{n,0}(e^{-ph}) = L_1^{n-1,0} A_{13} A_{33}^{-1} \qquad (2.16)$$

$$H_1^{mn}\left(e^{-ph}\right) =$$
$$= \left(A_{11}H_1^{n-1,m} - A_{13}\sum_{j=1}^{n}\sum_{k=j}^{m} \left\{ L_2^{j-1,K} H_1^{n-j,m-k} \right.\right. - \qquad (2.17)$$
$$\left. - L_3^{j-1,k} \sum_{r=0}^{n}\sum_{s=0}^{m} L_1^{r-j,s-k} H_1^{n-r-j,m-s-k} \right\} -$$
$$- \sum_{j=1}^{n}\sum_{k=0}^{m} \left(A_{12}L_1^{j-1,k} H_1^{n-j,m-k} + H_1^{j-1,k} L_1^{n-j,m-k} A_{12} \right) +$$
$$+ \sum_{j=1}^{n}\sum_{k=0}^{m-n} H_1^{j-1,k} \sum_{r=0}^{n}\sum_{s=0}^{m-n} L_1^{r-j,s-k} A_{13} L_3^{n-r-j,m-s-k} +$$
$$\left. + \sum_{j=0}^{n}\sum_{k=0}^{m-n} H_1^{j,k} A_{23} L_3^{n-j,m-k} - A_{13}L_3^{nm} \right)\left(A_{22} - A_{23}L_3^{00}\right)^{-1},$$
$$m > 0,\ n > 0,$$

with initial conditions

$$H_1^{00}(e^{-ph}) = \left[A_{12} - A_{13}A_{33}^{-1}A_{32}\right]\left[A_{22} - A_{23}A_{33}^{-1}A_{32}\right]^{-1},$$

$$H_2^{mn}\left(e^{-ph}\right) =$$
$$= \left(A_{11}H_2^{n-1,m-1} - A_{13}\sum_{j=1}^{n}\sum_{k=j}^{m} L_2^{j-1,k-1} H_2^{n-j,m-k} - \right.$$
$$- A_{12}\sum_{j=1}^{n}\sum_{k=1}^{m} L_1^{j-1,k-1} H_2^{n-j,m-k} + A_{12}H_3^{n,m-1} +$$
$$+ A_{13}\sum_{j=1}^{n}\sum_{k=j}^{m} L_3^{j-1,k-1} \sum_{r=0}^{n}\sum_{s=0}^{m} L_1^{r-j,s-k} H_1^{n-r-j,m-s-k} - \qquad (2.18)$$
$$- A_{13}\sum_{j=0}^{n}\sum_{k=j}^{m} L_3^{j,k-1} H_3^{n-j,m-k} -$$
$$- \sum_{j=1}^{n}\sum_{k=1}^{m-n} H_2^{j,k-1} L_3^{n-j,m-k} A_{23} -$$
$$\left. - \sum_{j=1}^{n}\sum_{k=j}^{m-n} H_2^{j-1,k-1} L_2^{n-j,m-k} A_{13} \right) A_{33}^{-1},\ m > 0,\ n > 0,$$

with initial conditions,

$$H_2^{00}\left(e^{-ph}\right) = A_{13}A_{33}^{-1},$$

$$H_2^{n,0}\left(e^{-ph}\right)=0, n=1,2,3,\ldots \qquad (2.19)$$

Considering the results of Lemma 2.1 and Lemma 2.2, and the linearity of (1.10), (1.11), (1.12) with respect to $H_1\left(\varepsilon_1,\mu,e^{-ph}\right), H_2\left(\varepsilon_1,\mu,e^{-ph}\right)$, $H_3\left(\varepsilon_1,\mu,e^{-ph}\right)$ it is possible to obtain the following Corollary.

***Corollary* 2.1.** *Let the assumptions of Lemmas* 2.1 *and* 2.2 *hold. The approximations,*

$$H_3\left(\varepsilon_1,\mu,e^{-ph}\right)= =\sum_{n=0}^{N}\varepsilon_1^n\sum_{m=0}^{\infty}\mu^m H_3^{nm}\left(e^{-ph}\right)+O\left(\varepsilon_1^{N+1}\right), \qquad (2.20)$$

$$H_1\left(\varepsilon_1,\mu,e^{-ph}\right)= =\sum_{n=0}^{N}\varepsilon_1^n\sum_{m=0}^{\infty}\mu^m H_1^{nm}\left(e^{-ph}\right)+O\left(\varepsilon_1^{N+1}\right), \qquad (2.21)$$

$$H_2\left(\varepsilon_1,\mu,e^{-ph}\right)= =\sum_{n=0}^{N}\varepsilon_1^n\sum_{m=0}^{\infty}\mu^m H_2^{nm}\left(e^{-ph}\right)+O\left(\varepsilon_1^{N+1}\right), \qquad (2.22)$$

where the terms, $H_3^{nm}\left(e^{-ph}\right)$, $H_2^{nm}\left(e^{-ph}\right)$, $H_1^{nm}\left(e^{-ph}\right)$ *of the asymptotic series can be found according to iterative schemes* (2.20)–(2.22) *are valid for all* $\varepsilon_1\in\left[0,\varepsilon_1^*\right)$, $\mu\in\left[0,\mu^*\right)$ *respectively such that* μ^* *and* ε_1^* *are satisfying* (2.2) *and* (2.9) *correspondingly.*

3 Results and Verifications

An illustrative example of TSPLTISD is considered for the application and the verification of the obtained results. Initially the system is solved using a standard tool with numerical approaches to obtain the exact subsystems for the considered TSPLTISD, on the MATLAB Simulink platform and then the obtained results are modeled so that they can be compared with the exact subsystems obtained.

Example. Let's consider an illustrative example

$$\begin{gathered}\dot{x}(t)=-x(t)-y(t)+u(t),\\ \varepsilon_1\dot{y}(t)=-x(t-1)-y(t),\\ \varepsilon_2\dot{z}(t)=-x(t-1)-z(t),\\ x,y,z\in\mathbb{R}^1,\end{gathered} \qquad (3.1)$$

with initial conditions:

$$x(0)=\varphi(0)=0.5,\ y(0)=2,$$
$$z(0)=1,\ x(\theta)=\varphi(\theta)=1,\ \theta\in[-1,0).$$

Considering the system (3.1) parameters in the form (1.2) can be denoted as:

$$\begin{gathered}h=1,\ n_1=n_2=n_3=r=1,\\ A_{11,0}=-1,\ A_{11,1}=0,\ A_{12}=-1,\ A_{13}=0,\ B_1=1,\\ A_{21,0}=0,\ A_{21,1}=-1,\ A_{22}=-1,\ A_{23}=0,\ B_2=0,\\ A_{31,0}=0,\ A_{31,1}=-1,\ A_{32}=0,\ A_{33}=-1,\ B_3=0.\end{gathered} \qquad (3.2)$$

Note that $\det A_{33}\neq 0$,

$$\det\left[A_{22}-A_{23}A_{33}^{-1}A_{32}\right]=-1\neq 0,$$

so, according to Lemmas 2.1, 2.2 and the Corollary, considered Sigularly Perturbed System (3.1) can be asymtotically decomposed with the use of the introduced approach. Matrix operator equations similar to (1.7)–(1.12) for (3.1) have the form (with $\lambda=e^{-ph}$):

$$L_2-\varepsilon_2L_2-\lambda-\mu L_3\lambda=0, \qquad (3.3)$$
$$L_3-\varepsilon_2L_2-\mu L_3=0, \qquad (3.4)$$
$$L_1-\lambda+\varepsilon_1L_1\left(L_1-1\right)=0, \qquad (3.5)$$
$$H_3-\varepsilon_2L_1H_3-\mu H_3=0, \qquad (3.6)$$
$$H_1-1+\varepsilon_1\left(L_1-1\right)H_1+\varepsilon_1H_1L_1=0, \qquad (3.7)$$
$$\varepsilon_2H_2(L_1-1)-\mu H_3+H_2=0. \qquad (3.8)$$

The first terms of expansions (2.3), (2.10) and (2.20)–(2.22) are,

$$L_1^{00}=\lambda,\ L_2^{00}=\lambda,\ L_3^{00}=0,\ H_1^{00}=1,\ H_2^{00}=0,\ H_3^{00}=0.$$

Evaluating (2.2), (2.9) using the induced norm $\|A\|_\infty\triangleq\max_i\sum_j\left|a_{ij}\right|$, $\left\|e^{-ph}\right\|=1$, the approximations (2.3), (2.10) and (2.15)–(2.17) are valid for all $\varepsilon_1\in\left[0,\varepsilon_1^*\right)$, $\mu\in\left[0,\mu^*\right)$, $\varepsilon_1^*=\frac{1}{2}$, $\mu^*=\frac{1}{3+2\sqrt{2}}$.

The *degenerate system* for (3.1) has the form,

$$\dot{x}_s(t)=-x(t)+x(t-h)+u(t),$$
$$x_s(0)=x_0=0,5,\ x_s(\theta)=\varphi(\theta)=1,\ \theta\in[-1,0).$$

ε_2 – *Boundary Layer System* has the form

$$\frac{d\hat{z}\left(\tau_{\varepsilon_2}\right)}{d\tau_{\varepsilon_2}}=-\hat{z}\left(\tau_{\varepsilon_2}\right),\ \hat{z}(0)=2,$$

where $\hat{z}(t)=z(t)+1$, $\tau_{\varepsilon_2}=\frac{t}{\varepsilon_2}$, $\frac{dx}{d\tau_{\varepsilon_2}}=0$, $\frac{dy}{d\tau_{\varepsilon_2}}=0$, $x=0,5$, $y=2$.

ε_1 – *Boundary Layer System* has the form

$$\frac{d\hat{y}\left(\tau_{\varepsilon_1}\right)}{d\tau_{\varepsilon_1}}=-\hat{y}\left(\tau_{\varepsilon_1}\right),\ \hat{y}(0)=3,$$

where $\hat{y}(t)=y(t)+1$, $\frac{dx}{d\tau_{\varepsilon_1}}=0$, $x=0,5$.

Similar to (9), (10), (14), (15), (19) and (20) of [14] and the initial iterations of matrix operators of asymptotic approximations of a decoupled system as discussed in [15], for (3.1) it has the form,

$$\begin{gathered}A_\xi^{00}\left(e^{-ph}\right)=-1+e^{-ph},\ A_\eta^{00}=-1,\ A_\beta^{00}=-1,\\ B_\xi^{00}=-1,\ B_\eta^{00}=0,\ B_\beta^{00}=0.\end{gathered} \qquad (3.9)$$

The 1st order approximation of the split system (3.1) has the form:

$$\begin{gathered}\dot{\xi}^0(t)=-\xi^0(t)+\xi^0(t-h)+u(t),\\ \varepsilon_1\dot{\eta}^0(t)=-\eta^0(t),\ \varepsilon_2\dot{\beta}^0(t)=-\beta^0(t),\end{gathered} \qquad (3.10)$$

with

$\xi^0(0) = x_0 = 0,5,\ \xi^0(\theta) = \varphi(\theta) = 1, \theta < 0,$

$\eta^0(0) = 3, \beta^0(0) = 2.$

The comparison of the exact solution of (3.1) under $u(t) \equiv 0$, $\varepsilon_1 = 0,01$, $\varepsilon_2 = 0,001$ and under $\varepsilon_1 = 0,1$, $\varepsilon_2 = 0,001$ with the solutions obtained on the basis of the 1st order approximations.

$$x(t) = \xi^0(t) + O(\mu),$$

$$y(t) = \eta^0(t) - L_1^{00}\xi^0(t) + O(\mu) =$$

$$= \eta^0(t) - \xi^0(t-h) + O(\mu),$$

$$z(t) = \beta^0(t) + \left(L_3^{00}L_1^{00} - L_2^{00}\right)\xi^0(t) - L_3^{00}\eta^0(t) + O(\mu) =$$

$$= \beta^0(t) - \xi^0(t-h) + O(\mu),$$

The simulation results obtained in the MATLAB Simulink has been presented in the figures 3.1–3.6.

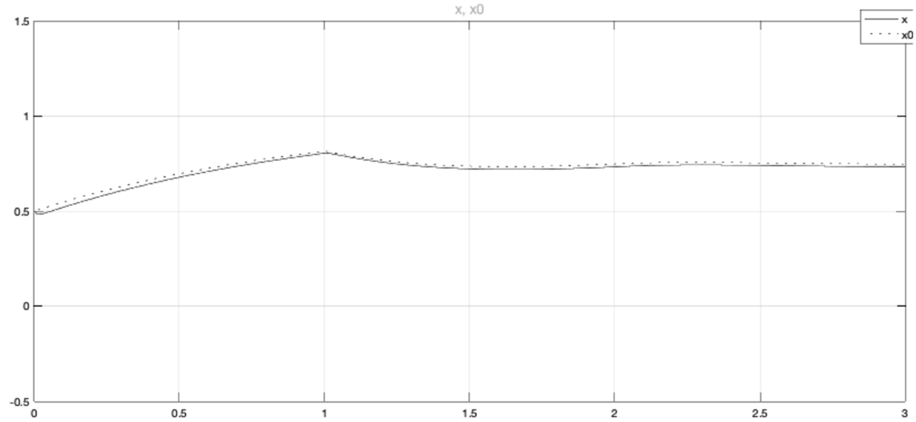

Figure 3.1 – x-component of exact (thick solid) and its asymptotic approximations of the 1st order (dotted line) under $\varepsilon_1 = 0,01, \varepsilon_2 = 0,001$

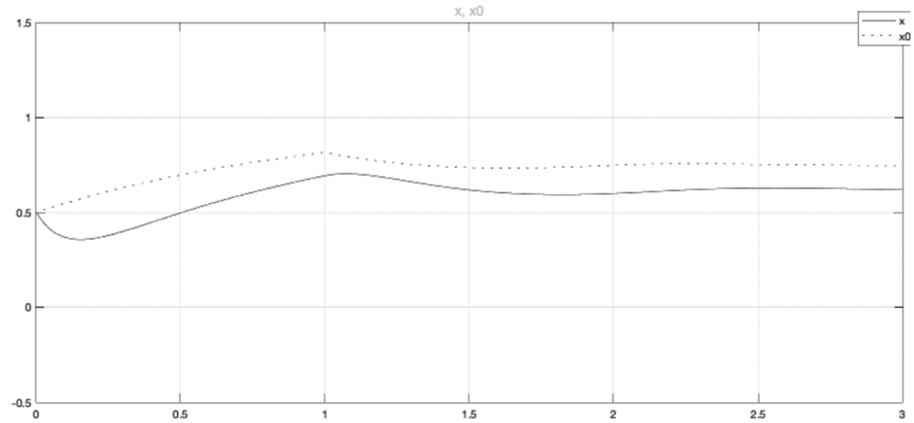

Figure 3.2 – x-component of exact (thick solid) and its asymptotic approximations of the 1st order (dotted line) under $\varepsilon_1 = 0,1,\ \varepsilon_2 = 0,001$

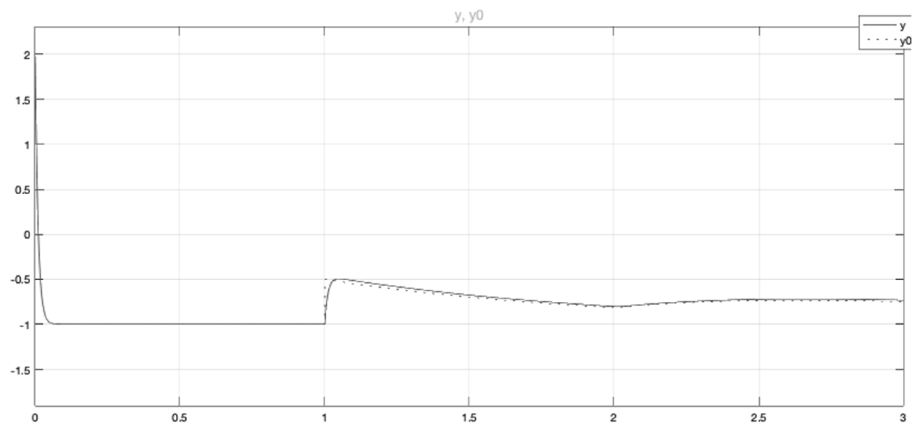

Figure 3.3 – y-component of exact (thick solid) and its asymptotic approximations of the 1st order (dotted line) under $\varepsilon_1 = 0,01, \varepsilon_2 = 0,001$

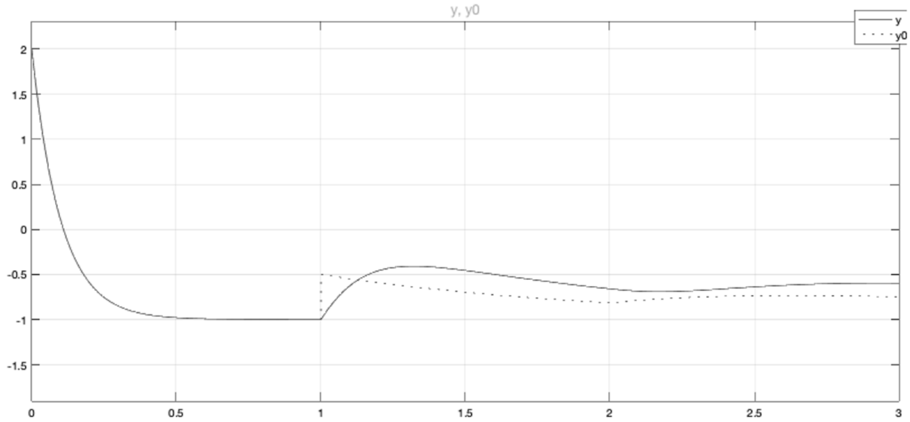

Figure 3.4 – y-component of exact (thick solid) and its asymptotic approximations of the 1st order (dotted line) under $\varepsilon_1 = 0,1,\ \varepsilon_2 = 0,001$

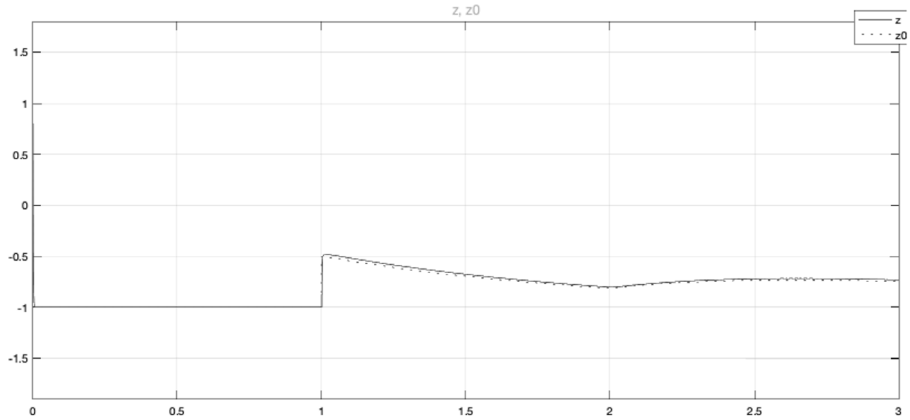

Figure 3.5 – z-component of exact (thick solid) and its asymptotic approximations of the 1st order (dotted line) under $\varepsilon_1 = 0,01,\ \varepsilon_2 = 0,001$

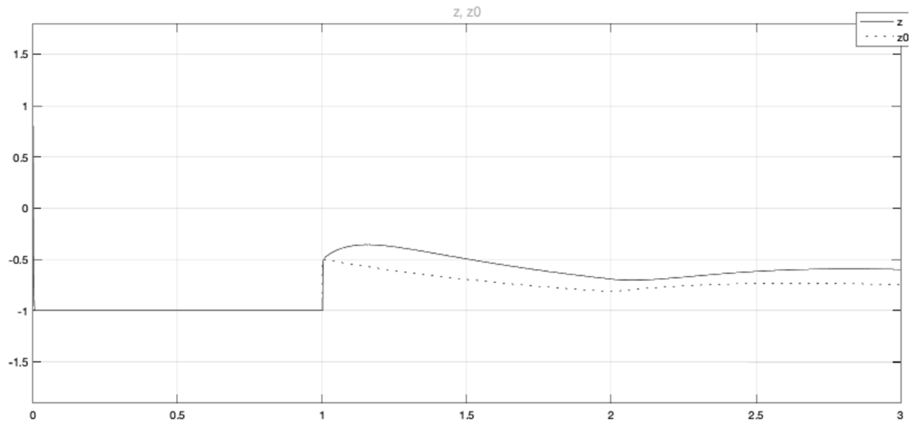

Figure 3.6 – z-component of exact (thick solid) and its asymptotic approximations of the 1st order (dotted line) under $\varepsilon_1 = 0,1,\ \varepsilon_2 = 0,001$

Conclusion

The asymptotic approximation discussed in this paper completely splits the Three-time-scale Singularly Perturbed Systems with multiple commensurate delays in the slow state variables (1.1), into their subsystems according to the tempo to any degree of accuracy in powers of the small parameters present in the system.

L and H need to satisfy the algebraic matrix equations (1.7)–(1.12) for the complete decoupling of the TSPLTISD into its subsystems respectively. Considering the fact that $f(D)$ is contractions mapping subjected to the constructed bounds of small parameters μ, ε_1, $f(D)$ has a unique solution which can be successively approximated starting with D_0.

Further by considering the concepts of the fixed point theorem for (2.7) and (2.13) and fact that $D = L - L_0$ and $D = H - H_0$ it can be concluded that the asymptotic approximations for the L and H matrixes hold, small parameter of the system subjected to the boundaries as defined by Lemma 2.1, 2.2, and Corollary 2.1. Thus, by Lemma 2.1, 2.2 and Corollary 2.1 on the reliability of the asymptotic approximations for the L, H matrixes, it can be considered that when the boundary requirements for the small parameters are met asymptotic approximations (2.3), (2.10) and (2.20)–(2.22) to be valid, and hence the subsystems (relatively faster, fast and slow variables) for the generalized TSPLTISD can be constructed in the form of asymptotic approximations.

Secondly, a sample TSPLTISD that satisfies the reliability boundaries for small parameters as stated in (2.7) and (2.13) is considered. With reference to the comparison results of the exact subsystems and the constructed 0^{th} degree asymptotic approximations for subsystems in (3.1) via Figures, 3.1–3.6 it can be concluded that asymptotic approximations are valid and accurately represents the subsystems in terms of the qualitative behavior of the subsystems.

Note that with reference to the simulation results shown in Figures 3.1–3.6 it can be concluded that, as the values of small parameters decrease, the accuracy of the approximation increases. As a rule, for the practice purposes, obtaining 0–1 approximations are sufficient.

But 1^{st} order approximation for the considered case (3.1) is not an accurate representation of the exact subsystem. And 2^{nd} order approximation and higher order approximations for the subsystems can be constructed that construct the subsystems more accurately. Based on (1.4), (1.5) and asymptotic approximations of L, H, higher approximations of system solutions could be obtained.

The use of the constructed asymptotic approximations of the decoupling transformation allows one to reduce the solution of a number of stability, control, and estimation problems for large systems with singular perturbations and delays to systems of lower dimension that are independent or regularly dependent on a small parameter. The decomposition algorithm can be implemented in software for computer algebra systems, the obtained results can be used to solve problems of analysis and synthesis of three time-scale linear stationary singularly perturbed systems with delay.

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