МАТЕМАТИКА

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О σ_i-ДЛИНЕ КОНЕЧНОЙ σ-РАЗРЕШИМОЙ ГРУППЫ

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ON THE σ_i-LENGTH OF A FINITE σ-SOLUBLE GROUP

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Пусть $\sigma = \{\sigma_i \mid i \in I\}$ некоторое разбиение множества всех простых чисел $\mathbb{P}$ и G – конечная группа. G называется σ-разрешимой, если каждый главный фактор H/K G – это σ_i-группа для некоторого $i = i(H/K)$. Мы доказываем следующую теорему.

Теорема. (i) Если G – π-отделимая группа, H – нильпотентная холлова π-подгруппа и E – π-дополнение группы G со свойством $EX = XE$ для некоторой подгруппы X в H такой, что $H' \le X \le \Phi(H)$, тогда $l_\pi(G) \le 1$.

(ii) Если G – σ-разрешимая группа и $\{H_1, \ldots, H_t\}$ – виландтов σ-базис группы G такой, что H_i перестановочна с $H_{j'}$ для всех i, j, тогда $l_{\sigma_i}(G) \le 1$ для всех i.

(iii) Если G – σ-разрешимая группа и $\{H_1, \ldots, H_t\}$ – виландтов σ-базис группы G такой, что H_i перестановочна с $\Phi(H_j)$ для всех i, j, тогда $l_{\sigma_i}(G) \le 1$ для всех i.

(iv) Если $l_\pi(G) \le 1$, то $QX = XQ$ для каждой характеристической подгруппы X группы H и любой силовской подгруппы Q в G такая, что $HQ = QH$.

(v) Если G – σ-разрешимая группа с $l_{\sigma_i}(G) \le 1$ для всех i и $\{H_1, \ldots, H_t\}$ является σ-базисом G, тогда каждая характеристическая подгруппа группы H_i перестановочна с каждой характеристической подгруппой группы H_j.

Ключевые слова: *конечная группа*, σ*-разрешимая группа*, π*-разделимая группа*, π*-длина холловой подгруппы*.

Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and G a finite group. G is said to be σ-soluble if every chief factor H/K of G is a σ_i-group for some $i = i(H/K)$. We prove the following

Theorem. (i) If G is π-separable, H is a nilpotent Hall π-subgroup and E a π-complement of G such that $EX = XE$ for some subgroup X of H such that $H' \le X \le \Phi(H)$, then $l_\pi(G) \le 1$.

(ii) If G is σ-soluble and $\{H_1, \ldots, H_t\}$ is a Wielandt σ-basis of G such that H_i permutes with $H_{j'}$ for all i, j, then $l_{\sigma_i}(G) \le 1$ for all i.

(iii) If G is σ-soluble and $\{H_1, \ldots, H_t\}$ is a Wielandt σ-basis of G such that of H_i permutes with $\Phi(H_j)$ for all i, j, then $l_{\sigma_i}(G) \le 1$ for all i.

(iv) If $l_\pi(G) \le 1$, then $QX = XQ$ each characteristic subgroup X of H and any Sylow subgroup Q of G such that $HQ = QH$.

(v) If G is σ-soluble with $l_{\sigma_i}(G) \le 1$ for all i and $\{H_1, \ldots, H_t\}$ is a σ-basis of G, then each characteristic subgroup of H_i permutes with each characteristic subgroup of H_j.

Keywords: *finite group*, σ*-soluble group*, π*-separable group*, π*-length*, *Hall subgroup*.

1 The concepts and results

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \ne j$.

A group G is said to be σ-*soluble* [1]–[3] if every chief factor H/K of G is a σ_i-group for some $i = i(H/K)$. In particular, G is said to be

π*-separable* if every chief factor of G is either a π-group or a π'-group.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* σ*-set* of G [1]–[3] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $i \in I$ such that $\sigma_i \cap \pi(G) \neq \varnothing$.

Note that if G is σ-soluble, then G has a σ-*basis* [1], that is, a complete Hall σ-set $\{H_1, \ldots, H_t\}$ such that $H_iH_j = H_jH_i$ for all i, j. Finally, recall that a *Wielandt* σ-basis of G is a σ-basis $\mathcal{H}$ of G such that every member of $\mathcal{H}$ is nilpotent.

Every π-separable group G has a series

$$1 = P_0(G) \leq M_0(G) < P_1(G) <$$
$$< M_1(G) < \cdots < P_t(G) \leq M_t(G) = G$$

such that

$$M_i(G)/P_i(G) = O_{\pi'}(G/P_i(G)) \quad (i = 0, 1, \ldots, t)$$

and $P_{i+1}(G)/M_i(G) = O_{\pi}(G/M_i(G)) \quad (i = 1, \ldots, t).$

The number t is called the π*-length* of G and denoted by $l_\pi(G)$ [4, p. 249].

In this note we prove the following

Theorem. (i) *If G is π-separable, H is a nilpotent Hall π-subgroup and E a π-complement of G such that $EX = XE$ for some subgroup X of H such that $H' \leq X \leq \Phi(H)$, then $l_\pi(G) \leq 1$.*

(ii) *If G is σ-soluble and $\{H_1, \ldots, H_t\}$ is a Wielandt σ-basis of G such that H_i permutes with $H_{j'}$ for all i, j, then $l_{\sigma_i}(G) \leq 1$ for all i.*

(iii) *If G is σ-soluble and $\{H_1, \ldots, H_t\}$ is a Wielandt σ-basis of G such that of H_i permutes with $\Phi(H_j)$ for all i, j, then $l_{\sigma_i}(G) \leq 1$ for all i.*

(iv) *If $l_\pi(G) \leq 1$, then $QX = XQ$ each characteristic subgroup X of H and any Sylow subgroup Q of G such that $HQ = QH$.*

(v) *If G is σ-soluble with $l_{\sigma_i}(G) \leq 1$ for all i and $\{H_1, \ldots, H_t\}$ is a σ-basis of G, then each characteristic subgroup of H_i permutes with each characteristic subgroup of H_j.*

***Corollary* 1.1.** *Suppose that G is p-soluble, and let P be a Sylow p-subgroup and E a p-complement of G. If $E\Phi(P) = \Phi(P)E$, then $l_p(G) \leq 1$.*

***Corollary* 1.2.** *Suppose that G is p-soluble, and let P be a Sylow p-subgroup. Then $l_p(G) \leq 1$ if and only if $QP' = P'Q$ for each Sylow subgroup Q of G such that $PQ = QP$.*

***Corollary* 1.3.** *Let G be soluble, and let P be a Sylow p-subgroup. Then $l_p(G) \leq 1$ if and only if $QP' = P'Q$ for each Sylow subgroup Q of G such that $PQ = QP$.*

***Corollary* 1.4** (Huppert [5, VI, Satz 6.11]). *Suppose that G is soluble, and let $P_1, \ldots, P_t$ be a Sylow basis of G. Then the following hold*:

(i) If $P_i'P_j = P_jP_i'$ for all i, j, *then* $l_p(G) \leq 1$ for all p;

(ii) *If $l_p(G) \leq 1$ for all p, then every characteristic subgroup of P_i permutes with each characteristic subgroup of P_j.*

2 Proof of the result

Let $\mathfrak{M}$ and $\mathfrak{H}$ be non-empty formations. Then the *Gaschütz product* $\mathfrak{M} \circ \mathfrak{H}$ of these formations is the class of all groups G such that $G^{\mathfrak{H}} \in \mathfrak{M}$. It is well-known that such an operation on the set of all non-empty formations is associative (W. Gaschütz). The symbol $\mathfrak{M}^t$ denotes the product of t copies of $\mathfrak{M}$.

We shall need the following well-known theorem of Gaschütz and Shemetkov [6, Corollary 7.13].

***Lemma* 2.1**. *The product of any two non-empty saturated formations is also a saturated formation.*

***Lemma* 2.2.** *The class $\mathcal{F}$ of all π-separable groups G with $l_\pi(G) \leq t$ is a saturated formation.*

Proof. It is not difficult to show that for any non-empty set $\omega \subseteq \mathbb{P}$ the class $\mathfrak{G}_\omega$ of all ω-groups is a saturated formation and that $\mathfrak{F} = (\mathfrak{G}_{\pi'} \circ \mathfrak{G}_\pi)^t \circ \mathfrak{G}_{\pi'}$. Hence $\mathfrak{F}$ is a saturated formation by Lemma 2.1. □

Proof of Theorem. (i) Suppose that this is false. Then $H \neq 1$.

(1) *For every minimal normal subgroup R of G we have $l_\pi(G/R) \leq 1$.*

Assume that this is false. Since G is π-separable, R is either a p'-group or a p-group. In the former case we have $H \simeq HR/R$, so

$$(HR/R)' = H'R/R$$

and $\Phi(H)R/R = \Phi(HR/R)$. Hence

$$(HR/R)' \leq XR/R \leq \Phi(HR/R),$$

where HR/R is a Hall π-subgroup of G/R. In the second case we have $R \leq H$, so

$$H'R/R = (H/R)' \leq$$
$$\leq XR/R \leq \Phi(H)R/R = \Phi(H/R).$$

Finally, ER/R is a π-complement of G/R and also we have

$$(ER/R)(XH/R) = EXH/R =$$
$$= XEH/R = (XR/R)(ER/R).$$

Therefore, the hypothesis holds for G/R and so we have (1) by the choice of G.

(2) *R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$. Hence $C_G(R) \leq R \leq O_\pi(G) \leq H$.*

The first assertion of (2) follows from Claim (1) and Lemma 2.2. Moreover, if R is a π'-group, then

$$l_\pi(G) = l_\pi(G/R) \leq 1.$$

Hence R is a π-group. Therefore, since $C_G(R)$ is normal in G, we have (2).

(3) *$R \cap X$ is normal in G.* Since

$$R \cap XE \le O_\pi(XE)$$

by Claim (1), $R \cap XE = R \cap X$ is normal in XE. Hence $E \le N_G(R \cap X)$. On the other hand, since $H' \le X \le H$ by hypothesis, X is normal in HP, so $R \cap X$ is normal in H. Therefore $R \cap X$ is normal in $G = HE$.

*Final contradiction fo*r (i). The minimality of R implies that either $R \cap X = 1$ or $R \cap X = R$. In the former case we have $X \le C_G(R)$ since X is normal in H and so $X = 1$. But then H is abelian since by hypothesis we have $H' \le X$. Therefore, since $C_G(R) \le R \le H$ by Claim (2), $R = H$ is normal in G. But then $l_\pi(G) \le 1$, which contradicts the choice of G. Therefore we have $R \le X \le \Phi(H)$ and so $R \le \Phi(G)$, contrary Claim (2). This final contradiction completes the proof of Part (i).

(ii) Since $\{H_1, \ldots, H_t\}$ is a Wielandt σ-basis of G, then H_i is nilpotent and

$$E = H_1 \cdots H_{i-1} H_{i+1} \cdots H_t$$

is a σ_i-complement of G. Moreover,

$$E\Phi(H_i) = \Phi(H_i)E$$

since $\Phi(H_i)H_j = H_j\Phi(H_i)$ for all j by hypothesis. Therefore $l_{\sigma_i}(G) \le 1$ for all i.

(iii) See the proof of (ii).

(iv) Suppose that this is false. Then $H \ne 1$ and Q is a q-group for some prime $q \notin \pi$. Suppose that $HQ < G$. Then $l_\pi(HQ) \le 1$ by Lemma 2.2, so $QX = XQ$ by the choice of G. Hence $G = HQ$.

Suppose that $O_{\pi'}(G) \ne 1$ and let R be a minimal normal subgroup of G contained in $O_{\pi'}(G)$. Then $R \le Q$ and $l_\pi(G/R) = l_\pi(G) \le 1$. Therefore the choice of G implies that

$$QX/R = (Q/R)(XR/R) = (X/R)(Q/R) = XQ/R$$

and so $QX = XQ$, which contradicts the choice of G. Therefore $O_{\pi'}(G) = 1$ and hence H is normal in G since $l_\pi(G) \le 1$ by hypothesis. But then X is normal in G since it is characteristic in H. Hence $QX = XQ$. This contradiction completes the proof of Part (ii).

(v) Lemma 2.2 implies that

$$l_\pi(H_iH_j) \le l_\pi(G) \le 1,$$

so in the case when $H_iH_j < G$, the choice of G implies that $V_iV_j = V_jV_i$. Therefore $H_iH_j = G$. Hence, by Part (iii), $V_iH_j = H_jV_i$ and $V_jH_i = H_iV_j$, so

$$V_iH_j \cap V_jH_i = V_i(H_j \cap V_jH_i) =$$
$$= V_i(H_j \cap H_i)V_j = V_iV_j = V_jV_i.$$

The theorem is proved.

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