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ПРОБЛЕМА ЦЕНТРА-ФОКУСА И ОТРАЖАЮЩАЯ ФУНКЦИЯ

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THE CENTER-FOCUS PROBLEM AND REFLECTING FUNCTION

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Дан один из способов применения отражающей функции для решения проблемы центра-фокуса.

Ключевые слова: *отражающая функция*, *дифференциальное уравнение*, *периодическое решение*, *проблема центра-фокуса.*

One way of application the reflecting function to the solution of the center-focus problem is given.

Keywords: *reflecting function*, *differential equation*, *periodic solution*, *centre-focus problem.*

Introduction

The center-focus problem is the oldest and important problem of the qualitative theory of differential equations. It originates from the works of H. Poincaré [1]. H. Poincaré put this problem and evaluated its significance. A.M. Liapunov [2, p. 136–252] suggested two methods for the solving this problem. The methods are connected with an infinite process of the calculating of so called Liapunov numbers $g_1, g_2, ...$. If one of the numbers $g_k \neq 0$ then the critical point under the consideration is focus. This point is center if and only if all $g_k = 0$ $(k \in N)$.

The interest to the center-focus problem was intensified in the Soviet Union and China after publication the book [3, p. 77–86]. The new approach appeared for the solving this problem was given in [4]. The algebraic insolvability of the problem in general compels researchers to consider the problem for the systems with the polynomial right-hand side. The summary of such investigation in the Soviet Union was made in the book [5].

The general consideration of the problem and corresponding results was given by V.I. Arnold and Iu.S. Iljashenko in the work [6].

1 The center-focus problem

The center-focus problem arises when we consider autonomous two dimension systems. The isolated critical point (x_0, y_0) (the rest-point) of the system is called center iff the point has a neighborhood which entirely consists of closed trajectories.

In the polar coordinate two-dimension differential system goes to equation

$$\frac{dr}{d\varphi} = \frac{rP(\varphi, r)}{Q(\varphi, r)}, \qquad (1.1)$$

with the rest point (equilibrium-point) $r = 0$.

This point $r = 0$ is a center iff all solutions $r = r(\varphi, \varphi_0, r_0)$ of the equation (1.1) with sufficiently small r_0 are 2π-periodic. Therefore the point $r = 0$ is center for (1.1) if and only if the reflecting function of the (1.1) $F(\varphi, r)$ [7] is 2π-periodic with respect to φ [8, p. 13] for sufficiently small r.

Thanks to that fact it is possible to look on the center-focus problem from the point of view of the reflecting function theory.

We put here some facts from the reflecting function theory which are necessary for the understanding this work and comprehension of further use the reflecting function for the problem. We set forth the facts here in accordance with [7], [8].

The reflecting function $F(t, x)$ for the system

$$\frac{dx}{dt} = X(t, x),\ t \in R,\ x \in R^n, \qquad (1.2)$$

with the general solution $\varphi(t; t_0, x_0)$ in the Cauchy form can be defined by formula $F(t, x) = \varphi(-t; t, x)$. The domain of the function consists of graphs of the solutions $x = \varphi(t; 0, x_0)$ which exist on symmetric intervals $(-\alpha_x; \alpha_{x_0})$.

We will assume that $X(t, x)$ is continuously differentiable function and mark the following properties of the reflecting function.

1^0. For every extendible solution $x(t)$ of the system (1.2) which exist on $[-\alpha; \alpha]$ the identity $F(t, x(t)) \equiv x(-t),\ \forall t \in [-\alpha; \alpha]$, is hold.

So the reflecting function connects the future state $x(t)$ of the system with its past state $x(-t)$ and vice versa. This property must serve as a definition of the reflecting function (RF).

2^0. For all (t, x) from the domain of the reflecting function $F(t, x)$ the identities

$$F(-t, F(t,x)) \equiv F(0,x) \equiv x$$

are true.

3^0. Differentiable function $F(t,x)$ is the reflecting function of the system (1.2) if and only if the function is the solution of the Cauchy problem

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} X(t,x) + X(-t,F) = 0,\ F(0,x) \equiv x. \quad (1.3)$$

We call this relation the main relation for the reflecting function.

4^0. If $X(t,x)$ continually differentiable and $X(t+2\omega) \equiv X(t,x)$, then Poincaré map $\varphi(\omega;-\omega,x)$ of the system (1.2) over $[-\omega,\omega]$ is given by formula $F(-\omega,x)$, where $F(t,x)$ is reflecting function of the system. In this case the solution $x = \varphi(t;-\omega,x_0)$ of the system (1.2) will be 2ω-periodic if and only if the solution is extendible on $[-\omega;\omega]$ and $F(-\omega,x_0) = x_0$.

5^0. The reflecting function $F(t,x)$ of the system (0.2) for which $X(-t,x) + X(t,x) \equiv 0$ is given by formula $F(t,x) \equiv x$. If in addition $X(t,x)$ is 2ω-periodic with respect to t then every solution of the system (1.2) which is extendible on $[-\omega,\omega]$ is 2ω-periodic.

6^0. Every twice continuously differentiable function $F(t,x)$, $F: D \to R^n$, where D contains the hyperplane $x=0$ and F has the property $F(-t,F(t,x)) \equiv F(0,x) \equiv x$ is reflecting function of the system

$$\frac{dx}{dt} = -\frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^{-1} \frac{\partial F}{\partial t} =: S(t,x) \quad (1.4)$$

and every system of the form

$$\frac{dx}{dt} = -\frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^{-1} \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial x}\right)^{-1} R(t,x) - \\ -R(-t,F), \quad (0.5)$$

where $R(t,x)$ is any continuously differentiable vector-function.

All such systems form the class of equivalent systems, which is characterized by reflecting function $F(t,x)$.

The system (0.4) is called simple system of the class. All autonomous systems are simple systems.

7^0. If system $\frac{dx}{dt} = X(t,x)$ is equivalent an autonomous system, then this autonomous system is

$$\frac{dx}{dt} = X(0,x).$$

Sometimes it is very convenient to use the following

***Statement* 1.1**. [9], [10, p. 171]. *Suppose that* $\Delta_i(t,x)$ $(i = 1;m)$ *are solution of the system*

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X(t,x) - \frac{\partial X}{\partial x}(t,x)\Delta = 0.$$

Then all systems

$$\frac{dx}{dt} = X(t,x) + \sum_{i=1}^{m} \alpha_i(t)\Delta_i(t,x),$$

where $\alpha_i(t)$ *are scalar odd continuous functions, are equivalent* (*i. e. they have the same reflecting function which coincides with the reflecting function of the system* $\frac{dx}{dt} = X(t,x)$).

This statement and theorem from [11] are very important for the solvability of center-focus problem.

2 The main results

We consider the equation

$$\frac{dr}{d\varphi} = \frac{A_1 r + A_2 r^2 + \ldots + A_n r^n}{1 + B_2 r + B_3 r^2 + \ldots + B_k r^{k-1}}, \quad (2.1)$$

where $A_i = A_i(\varphi)$ and $B_i = B_i(\varphi)$ are continues 2π-periodic functions.

We assume that this equation obtained from the two dimensional differential autonomous system for which the center-focus problem is arising. Variables φ and r are coordinates of polar system. The point $(x,y) = (0,0)$ or $r = 0$ is the equilibrium point for which the center-focus problem is arising. The function $r(\varphi) \equiv 0$ is the solution of (2.1). The solutions $r(\varphi;0,r_0)$ of the equation (1.1) for the sufficiently small r_0 are extendible on $[-\pi;\pi]$ .due to the theorem of continuous dependence solutions on the initial conditions dates [3, p. 13].

Then in accordance with the property 4^0 the equation (2.1) will have center at $r=0$ if and only if for the reflecting function $F(\varphi,r)$ of the (2.1) the identity $F(\pi,r) \equiv r$ or if and only if the reflecting function will be 2π-periodic [8, p. 66].

Thus the center-focus problem for the equation (2.1) reduces itself to the question: “When equation (2.1) has 2π-periodic with respect to φ reflecting function or when equation (2.1) is equivalent to any different equation of the form (2.1) which has center at $r=0$ ”.

***Theorem* 2.1.** *Suppose*

1. $U = U(\varphi,r) = m_1(\varphi)r + m_2(\varphi)r^2 + \ldots + m_k(\varphi)r^k$ *is a polynomial with respect to* r *with continuously differentiable and* 2π*-periodic coefficients* $m_1(\varphi)$ $(i = \overline{1;k})$ *and* $m_1(\varphi) \neq 0$ *for all* φ.

2. $R(\varphi,U)$ *is polynomial with respect to* U *with the continuous even* 2π*-periodic coefficients and* $R(\varphi;0) \neq 0$ $\forall \varphi \in R$.

3. $S(\varphi,U)$ *is polynomial with respect to* U *with continues* 2π*-periodic odd coefficients and* $S(\varphi,0) \equiv 0$.

Then the reflecting function $F(\varphi, r)$ *of the equation*

$$\frac{dr}{d\varphi} = \frac{S(\varphi, U) - U'_\varphi R(\varphi, U)}{U'_r(\varphi, r) R(\varphi, U)} \qquad (2.2)$$

is given by formula $U(-\varphi, F) = U(\varphi, r)$ *and point* $r = 0$ *is a center for the equation* (2.2).

Proof. We shall find the complete derivative of $U(\varphi, r)$ along with the solutions of (2.2)

$$\frac{dU}{d\varphi} = \frac{\partial U}{\partial \varphi} + \frac{\partial U}{\partial r}\frac{dr}{d\varphi} =$$

$$= U'_\varphi + \frac{S(\varphi, U) - U'_\varphi R(\varphi, U)}{R(\varphi, U)} = \frac{S(\varphi, U)}{R(\varphi, U)}.$$

So $\frac{dU}{d\varphi} = \frac{S(\varphi, U)}{R(\varphi, U)}$ is odd function with respect to U. Therefore for every solution $r(\varphi)$ of the equation (2.2) $U(\varphi, r(\varphi))$ is even 2π-periodic function [8, p. 13], [10, p. 65], that is

$$U(-\varphi, r(-\varphi)) \equiv U(\varphi, r(\varphi))$$

or $U(-\varphi, F(\varphi, r(\varphi))) \overset{\varphi}{\equiv} U(\varphi, r(\varphi))$ for every solution $r(\varphi)$. It means $U(-\varphi, F(\varphi, r)) \overset{\varphi, r}{\equiv} U(\varphi, r)$. In this identity $U(\varphi, 0) \equiv 0$ and $\frac{\partial U}{\partial r}(\varphi, 0) \equiv m_1(\varphi) \neq 0.$ Therefore in accordance with the implicit function theorem [12, p. 488] the reflecting function $F(\varphi, r)$ exist differentiable and 2π-periodic with respect to φ.

The theorem is proved. So point $r = 0$ is a center. □

Now it is important to know when the equation (2.1) can be written in the form (2.2). First we consider the case when $R(\varphi, r) \equiv 1$ in (2.2). In this case (2.2) has the form

$$\frac{dr}{d\varphi} = \\ = \frac{s(\varphi, U) - m'_1(\varphi) r - m'_2(\varphi) r^2 - \ldots - m'_k(\varphi) r^k}{m_1(\varphi) + 2m_2(\varphi) r + \ldots + k m_k(\varphi) r^{k-1}}, \qquad (2.3)$$

It follows from this that to write the equation (2.1) in the form (2.3) we must multiply the numerator and the denominator of the right hand-side of the equation (2.1) on the function

$$m_1(\varphi) = \exp\left(-\int_0^\varphi A_1(\tau) d\tau\right).$$

In such a way the equation (2.1) gets the form

$$\frac{dr}{d\varphi} = \frac{D_1(\varphi) r + D_2(\varphi) r^2 + \ldots + D_n(\varphi) r^n}{m_1(\varphi) + 2m_2(\varphi) r + \ldots + k m_k(\varphi) r^{k-1}}, \quad (2.4)$$

where

$$m_1(\varphi) := \exp\left(-\int_0^\varphi A_1(\tau) d\tau\right). \qquad (2.5)$$

Here $A_1(\varphi)$ is the function from the (2.1). In form (2.4) we can rewrite every equation (2.1) and

$$m_1(\varphi) := \exp\left(-\int_0^\varphi A_1(\tau) d\tau\right) \neq 0.$$

In edition we will assume that function

$$m_1(\varphi) := \exp\left(-\int_0^\varphi A_1(\tau) d\tau\right)$$

is 2π-periodic function. If it is not 2π-periodic then the Liapunov number

$$g_1 := \frac{1}{2\pi}\int_0^{2\pi} A_1(\tau) d\tau \neq 0$$

and the equilibrium $r = 0$ is focus [5, p. 7] and the center-focus problem is solved.

***Theorem* 2.2.** *Suppose that for a differentiable function* $F(\varphi, r)$, $F(0, r) \equiv r$, *the following identities*

$$\sum_{i=1}^k m_i(-\varphi) F^i = \sum_{i=1}^k m_i(\varphi) r^i. \qquad (2.6)$$

$$\sum_{i=1}^k m'_i(-\varphi) F^i + \sum_{i=1}^k m'_i(\varphi) r^i + \\ + \sum_{i=1}^n D_i(\varphi) r^i + \sum_{i=1}^n D_i(-\varphi) F^i \equiv 0. \qquad (2.7)$$

are correct. Then the reflecting function of the equation (2.4) *is given by formula* (2.6) *and the equilibrium* $r = 0$ *for* (2.4) *is the center.*

Proof. From the identity (2.6) we find

$$\frac{\partial F}{\partial r} = \frac{\sum_{i=1}^k i m_i(\varphi) r^{i-1}}{\sum_{i=1}^k i m_i(-\varphi) F^{i-1}},$$

$$\frac{\partial F}{\partial \varphi} = \frac{\sum_{i=1}^k m'_i(\varphi) r^i + \sum_{i=1}^k m'_i(-\varphi) F^i}{\sum_{i=1}^k i m_i(-\varphi) F^{i-1}}.$$

Then the main relation for the reflecting function (1.3) we can write as follows

$$\frac{\sum_{i=1}^k m'_i(\varphi) r^i + \sum_{i=1}^k m'_i(-\varphi) F^i}{\sum_{i=1}^k i m_i(-\varphi) F^{i-1}} +$$

$$+ \frac{\sum_{i=1}^k i m_i(\varphi) r^{i-1}}{\sum_{i=1}^k i m_i(-\varphi) F^{i-1}} \times \frac{\sum_{r=1}^n D_i(\varphi) r_i}{\sum_{i=1}^k i m_i r^{i-1}} + \frac{\sum_{i=1}^n D_i(-\varphi) F^i}{\sum_{i=1}^k i m_i(-\varphi) F^{i-1}} \equiv 0.$$

This identity is really true due to (2.7).

So the reflecting function of (2.4) is given by (2.6) and all solutions of (2.4) are 2π-periodic. The theorem 2.2 is true. □

We consider now the equation (1.1). Let

$$m_1(\varphi) = \exp\left(-\int_0^\varphi A_1(\tau) d\tau\right)$$

and

$$U(\varphi,r)=m_1(\varphi)r+m_1(\varphi)\beta_2(\varphi)r^2+\ldots+$$
$$+m_1(\varphi)\beta_r(\varphi)r^k=:m_1r+m_2r^2+\ldots+m_kr^k.$$

We suppose that there exists function

$$v(\varphi,r):=m_1r+m_2r^2+\ldots+m_kr^k$$

and even 2π-periodic continuous functions $n_2(\varphi),n_3(\varphi),\ldots,n_l(\varphi)$ such that

$$U(\varphi,r)\equiv v+n_2(\varphi)v^2+n_3(\varphi)v^3+\ldots+n_l(\varphi)v^l.$$

In this case if reflecting function $F(\varphi,r)$ of the equation (2.1) is given by $U(-\varphi,F)=U(\varphi,r)$ then this $F(\varphi,r)$ is also determined by $v(-\varphi,F)\equiv v(\varphi,r)$. Therefore

$$\frac{\partial F}{\partial r}=\frac{v'_r(\varphi,r)}{v'_r(-\varphi,F)},\ \frac{\partial F}{\partial\varphi}=\frac{v'_\varphi(\varphi,r)+v'_\varphi(-\varphi,F)}{v'_r(-\varphi,F)}$$

and the main relation for the reflecting function we can write in the form

$$\frac{v'_\varphi(\varphi,r)+v'_\varphi(-\varphi,F)}{v'_r(-\varphi,F)}+$$
$$+\frac{v'_r(\varphi,r)}{v'_r(-\varphi,F)}\frac{A_1(\varphi)R+\ldots+A_n(\varphi)r^n}{1+\beta_2(\varphi)r+\ldots+\beta_k(\varphi)r^{k-1}}+ \qquad (1.8)$$
$$+\frac{A_1(-\varphi)F+A_2(-\varphi)F^2+\ldots+A_n(-\varphi)F^n}{1+\beta_2(-\varphi)F+\ldots+\beta_k(-\varphi)F^{k-1}}\equiv 0.$$

Thus we come to the

***Corollary* 2.1.** *If the relation* $v(-\varphi,F)=v(\varphi,r)$ *induces* (2.8), *then the reflecting function of the equation* (2.1) *is given by* $v(-\varphi,F)=v(\varphi,r)$.

***Example* 2.1.** We consider the equation

$$\frac{dr}{d\varphi}=r^2\frac{(1-r\sin\varphi)^2\sin\varphi+[1-2r(1-r\sin\varphi)]\cos\varphi}{(1-2r\sin\varphi)[1-2r(1-r\sin\varphi)\cos\varphi}.$$

For this equation $A_1(\varphi)\equiv 0$ and therefore

$$m_1=\exp\left(-\int_0^\varphi A_1(\tau)d\tau\right)\equiv 1.$$

So this equation has already the form (2.3). Then the function

$$U(\varphi,r)=\int_0^r\frac{\partial U}{\partial r}dr=$$
$$=\int_0^r(1-2\tau\sin\varphi)[1-2\tau(1-\tau\sin\varphi)\cos\varphi]d\tau=$$
$$=\int_0^r[1-2\tau(\tau-\tau^2\sin\varphi)\cos\varphi](\tau-\tau^2\sin\varphi)'_r\,dr=$$
$$=(r-r^2\sin\varphi)^2\cos\varphi.$$

It means that

$$U(\varphi,r)=v-v^2\cos\varphi,$$

where $v=r-r^2\sin\varphi$. So we expect that reflecting function of the equation will be given by relation $F+F^2\sin\varphi=r-r^2\sin\varphi$. Therefore

$$\frac{\partial F}{\partial r}=\frac{1-2r\sin\varphi}{1+2F\sin\varphi},\ \frac{\partial F}{\partial\varphi}=\frac{-(r^2+F^2)\cos\varphi}{1+2F\sin\varphi}.$$

We can rewrite the equation in the form

$$\frac{dr}{d\varphi}=\frac{v^2\sin\varphi}{1-2v\cos\varphi}+\frac{r^2\cos\varphi}{1-2r\sin\varphi}\times\frac{dr}{d\varphi}=$$
$$=r^2\left(\frac{(1-r\sin\varphi)^2(1-2r\sin\varphi)\sin\varphi}{(1-2v\cos\varphi)(1-2r\sin\varphi)}+\right.$$
$$\left.+\frac{\cos\varphi-2r(1-r\sin\varphi)\cos^2\varphi}{(1-2v\cos\varphi)(1-2r\sin\varphi)}\right).$$

We can write the main relation for the reflecting function F for which $v(-\varphi,F)\equiv v(\varphi,r)$ as follows

$$\frac{-(r^2+F^2)\cos\varphi}{1+2F\sin\varphi}+\frac{1-2r\sin\varphi}{1+2F\sin\varphi}\times$$
$$\times\left(\frac{v^2\sin\varphi}{1-2v\cos\varphi}+\frac{r^2\cos\varphi}{1-2r\sin\varphi}\right)+$$
$$+\frac{-v^2\sin\varphi}{1-2v\cos\varphi}+\frac{F^2\cos\varphi}{1+2F\sin\varphi}=0.$$

This relation is identity, as the reader can see. So the function $F=F(\varphi,r)$ is reflecting function of the equation under consideration and the point $r=0$ is center.

The equation from example 2.1. is equivalent to any equation of the form

$$\frac{dr}{d\varphi}=\frac{r^2\cos\varphi+S(\varphi,v)}{1-2r\sin\varphi},$$

where $S(-\varphi,v)\equiv S(\varphi,v)$.

The next theorem allow us to replace the consideration of one equation by another.

***Theorem* 2.3.** *Suppose the polynomial* $S(\varphi,r)$ *with respect to* r *and with differentiable coefficients satisfy the following identities*

$$\frac{\partial S}{\partial\varphi}Q-S\frac{\partial Q}{\partial\varphi}+\frac{\partial S}{\partial r}P-\frac{\partial P}{\partial r}S\equiv 0.$$

Then for every continuous and odd function $\alpha(\varphi)$ *the reflecting function of the equation* $\frac{dr}{d\varphi}=\frac{P(\varphi,r)}{Q(\varphi,r)}$ *coincides with the reflecting function of the equation*

$$\frac{dr}{d\varphi}\frac{P(\varphi,r)+\alpha(\varphi)S(\varphi,r)}{Q(\varphi,r)}.$$

For the proof it is sufficient to use the property 7^0 with $\Delta=\frac{S(\varphi,r)}{Q(\varphi,r)}$.

***Example* 2.2.** The equation

$$\frac{dr}{d\varphi}=\frac{P_2(\varphi)r^2+P_3(\varphi)r^3}{1+(cq(\varphi)+c_0)r+q(\varphi)r^2},$$

where P_2,P_3,q are continuous 2π-periodic functions and c,c_0 – constants, is equivalent to the equation

$$\frac{dr}{d\varphi}=\frac{P_2(\varphi)r^2+P_3(\varphi)r^3+\alpha(\varphi)[r^2+cq(\varphi)r^3]}{1+(cq(\varphi)+c_0)r+q(\varphi)r^2},$$

where $\alpha(\varphi)$ is odd continuous. So when $P_2(\varphi) = -\alpha(\varphi)$ or when $P_3(\varphi) = -c\alpha(\varphi)q(\varphi)$ the investigation of the first equation is very simple.

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