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КОНЕЧНЫЕ ГРУППЫ С ОБОБЩЕННО σ-СУБНОРМАЛЬНЫМИ И σ-ПЕРЕСТАНОВОЧНЫМИ ПОДГРУППАМИ

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FINITE GROUPS WITH GENERALIZED σ-SUBNORMAL AND σ-PERMUTABLE SUBGROUPS

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На протяжении всей статьи все группы конечны, и *G* всегда обозначает конечную групу. Мы говорим, что подгруппа *H* из *G* почти модульна в *G*, если *A* нормальна в *G* или $H_G \neq H^G$ и каждый главный фактор H/K группы *G* между H_G и H^G является почти центральным в *G*, то есть $|H/K||GC_G(H/K)|$ делит *pq* для некоторых простых чисел *p* и *q*. Мы говорим, что подгруппа *A* группы *G* является:
(i) *почти* σ*-субнормальной* в *G*, если $A=\langle L,T\rangle$, где *L* является почти модульной подгруппой и *T* является σ-субнормальной подгруппой в *G*;
(ii) *почти* σ*-перестановочной* в *G*, если $A=\langle L,T\rangle$, где *L* – почти модульная подгруппа и *T* является σ-перестановочной подгруппой *G*.
(iii) *слабо* σ*-перестановочной* в *G*, если в *G* имеется почти σ-перестановочная подгруппа *S* и подгруппа *T* такая, что $G=HT$ и $H\cap T\leq S\leq H$.
В данной статье изучаются конечные группы с некоторыми системами почти σ-субнормальных, почти σ-перестановочных и слабо σ-перестановочных подгрупп. Обобщены некоторые известные результаты.

Ключевые слова: *конечная группа*, *n-максимальная подгруппа*, *почти* σ*-субнормальная подгруппа*, *почти* σ*-перестановочная подгруппа*, *почти* σ*-нильпотентная группа*.

Throughout this paper, all groups are finite and *G* always denotes a finite group. We say that a subgroup *H* of *G* is nearly modular in *G* if either *A* is normal in *G* or $H_G \neq H^G$ and every chief factor H/K of *G* between H_G and H^G is nearly central in *G*, that is, $|H/K||G/C_G(H/K)|$ divides *pq* for some primes *p* and *q*. We say that a subgroup *A* of *G* is:
(i) *nearly* σ*-subnormal* in *G* if $A=\langle L,T\rangle$, where *L* is a nearly modular subgroup and *T* is a σ-subnormal subgroup of *G*;
(ii) *nearly* σ*-permutable* in *G* if $A=\langle L,T\rangle$, where *L* is a nearly modular subgroup and *T* is a σ-permutable subgroup of *G*.
(iii) *weakly* σ*-permutable* in *G* if there are a nearly σ-permutable subgroup *S* and a subgroup *T* of *G* such that $G=HT$ and $H\cap T\leq S\leq H$.
In the given paper, we study finite groups with some systems of nearly σ-subnormal, nearly σ-permutable and weakly σ-permutable subgroups. Some known results are generalized.

Keywords: *finite group*, *n-maximal subgroup*, *nearly* σ*-subnormal subgroup*, *nearly* σ*-permutable subgroup*, σ*-nearly nilpotent group.*

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1 The concepts and results

Throughout this paper, all groups are finite and *G* always denotes a finite group.

We say that: a chief factor H/K of *G* is *nearly central* in *G* if $|H/K||G/C_G(H/K)|$ divides *pq* for some primes *p* and *q*; a subgroup *H* of *G* is *nearly modular* in *G* if either *A* is normal in *G* or $H_G \neq H^G$ and every chief factor of *G* between H_G and H^G is nearly central in *G*.

In what follows, σ is some partition of $\mathbb{P}$ [1], that is, $\sigma=\{\sigma_i \mid i\in I\}$, where $\mathbb{P}=\bigcup_{i\in I}\sigma_i$ and $\sigma_i\cap\sigma_j=\varnothing$ for all $i\neq j$.

We say that a *chief* factor H/K is: σ*-central* in *G* if $|H/K||G/C_G(H/K)|$ is σ-primary; *nearly* σ*-central* in *G* if H/K is *either* σ-central or nearly central in *G*.

A set $\mathcal{H}$ of subgroups of *G* is a *complete Hall* σ*-set* of *G* [2], [3] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of *G* for some $\sigma_i\in\sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of *G* for all *i*; *G* is said to be σ-full if *G* possesses a complete Hall σ-set.

A subgroup *A* of *G* is called [1]: σ*-subnormal* in *G* [1] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i / (A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; σ-permutable in G if G is σ-full and A permutes with all Hall σ_i-subgroups of G for all i.

The σ-subnormal and σ-permutable subgroups proved to be very useful and found many applications in the study of various classes of generalized solvable groups (see, for example, the papers [1]–[13]). In this paper, we consider the following generalizations of these two concepts.

***Definition* 1.1.** We say that a subgroup A of G is:

(i) *nearly* σ*-subnormal* (respectively *nearly subnormal*) in G if $A = \langle L, T \rangle$, where L is a nearly modular subgroup and T is a σ-subnormal (respectively subnormal) subgroup of G;

(ii) *nearly* σ*-permutable* (respectively nearly S-permutable) in G if $A = \langle L, T \rangle$, where L is a nearly modular subgroup and T is a σ-permutable (respectively an S-permutable) subgroup of G;

(iii) *weakly* σ*-permutable* (respectively weakly S-permutable) in G if there are a nearly σ-permutable (respectively a nearly S-permutable) subgroup S and a subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$.

***Example* 1.2.** Let p, q, r, t be distinct primes, where q divides $p-1$ and T divides $r-1$. Let $V = Q \rtimes C_p$, where q is a simple $\mathbb{F}_q C_p$-module which is faithful for C_p $C_r \rtimes C_t$ a non-abelian group of order rt.

(i) Let $G = (Q \rtimes C_p) \times (C_r \rtimes C_t)$. Let B be a subgroup of order q in q. Then $B < Q$ since $p > q$. Let $H = \langle C_r, B \rangle$. Then A is nearly modular in G, so H is nearly subnormal in G. Assume that H is nearly modular in G. Then $B = H \cap (Q \rtimes C_p)$ is nearly modular in $(Q \rtimes C_p)$ by Lemma 2.8 (4) in [15]. Hence q is cyclic. This contradiction shows that H is not nearly modular in G. Similarly, if H is subnormal in G, then $C_t = H \cap (C_r \rtimes C_t)$ is subnormal in $C_r \rtimes C_t$ and so C_t is normal in $C_r \rtimes C_t$. But then $C_r \rtimes C_t$ is abelian. This contradiction shows that H is not subnormal in G.

(ii) Now, let p be a simple $\mathbb{F}_p V$-module which is faithful for V and $G = (P \rtimes (Q \rtimes C_p)) \times (C_r \rtimes C_t)$. Since q divides $p-1$, pq is supersoluble. Hence for some normal subgroup B of pq we have $1 < B < P$. Then for every Sylow p-subgroup G_p of G we have $B \leq P \leq G_p$, so $BG_p = G_p = G_p B$. On the other hand, for every Sylow q-subgroup Q^x of G we have $Q^x \leq PQ$, so $BQ^x = Q^x B$. Hence B is S-permutable in G. It is clear that C_t is nearly modular in G. Then $H = \langle A, B \rangle$ is nearly S-permutable in G. Moreover, H is neither nearly modular nor S-permutable in G.

We say that G is: *nearly* σ*-nilpotent* if every non-frattini chief factor H/K of G (that is, $H/K \nleq \Phi(G/K)$) is nearly σ-central in G, *srtongly supersoluble* if G is supersoluble and G induces on any its chief factor H/K an automorphism group of square free order.

Recall that if

$$M_n < M_{n-1} < \ldots < M_1 < M_0 = G, \qquad (1.1)$$

where M_i is a maximal subgroup of M_{i-1} for all $i = 1, \ldots, n$, then the chain (1.1) is said to be a *maximal chain of G of length n* and M_n $(n > 0)$, is an *n-maximal subgroup* of G.

Our first observation is the following

***Theorem* 1.3.** (i) *If in every maximal chain* $M_3 < M_2 < M_1 < M_0 = G$ *of G, of length* 3, *at least one of the subgroups* M_3, M_2, *or* M_1 *is nearly* σ*-subnormal in G, then G is* σ*-soluble.*

(ii) *If every 2-maximal subgroup of G is* σ*-permutable, then G is either* σ*-nilpotent or supersoluble.*

(iii) *If every 2-maximal subgroup of G is nearly S-permutable in G, then G is a nearly nilpotent group. Hence G is strongly supersoluble.*

***Corollary* 1.4** (Spencer [16]). *If in every maximal chain* $M_3 < M_2 < M_1 < M_0 = G$ *of G, of length* 3, *at least one of the subgroups* M_3, M_2, *or* M_1 *is subnormal in G, then G is soluble.*

***Corollary* 1.5** (Agrawal [17]). *If every* 2*-maximal subgroup of G is S-permutable in G, then G is supersoluble*.

***Corollary* 1.6** (Huppert [18]). *If every* 3*-maximal subgroup of G is normal in G, then G is soluble.*

***Corollary* 1.7** (Guo, Skiba in [13]). *If in every maximal chain* $M_3 < M_2 < M_1 < M_0 = G$ *of G, of length* 3, *at least one of the subgroups* M_3, M_2, *or* M_1 *is* σ*-subnormal in G, then G is* σ*-soluble*.

A subgroup M of G is called *modular* if M is a modular element (in the sense of Kurosh [19, 2, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is,

(i) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$, and

(ii) $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

From Theorem 5.2.5 in [19] it follows that every modular subgroup of G is also nearly modular in G. Hence we get from Theorem 1.3 (iii) the following known result.

***Corollary* 1.8** (Schmidt [20]). *If every* 2*-maximal subgroup M of G is modular*, *then G is nearly nilpotent*.

***Theorem* 1.9.** *Suppose that G is not σ-nilpotent. Then every maximal chain of length* 2 *in G includes a proper nearly σ-subnormal subgroup of G if and only if G is either nearly σ-nilpotent or a Schmidt group* (*that is*, *a non-nilpotent group all of which subgroups are nilpotent*) *with abelian Sylow subgroups*.

In the case when $\sigma = \{\{2\},\{3\},\dots\}$ we get from Theorem 1.9 the following known results.

***Corollary* 1.10** (See Theorem A in [21]). *Suppose that G is not nilpotent. Then every maximal chain of length* 2 *in G includes a proper subgroup H of G of the form* $M = \langle A, B\rangle$, *where A is modulart and B is subnormal in G*, *if and only if G is either nearly nilpotent or a Schmidt group with abelian Sylow subgroups*.

***Corollary* 1.11** (Schhmidt [20]). *If every* 2*-maximal subgroup M of G is modular*, *then G is nearly nilpotent.*

Note that Theorems 1.3 and 1.9 are the basis in the proofs of all other results of this paper. In particular, being based on these results we obtain the following result.

***Theorem* 1.12.** *Suppose that G is soluble and every n-maximal subgroup of G is nearly S-permutable in G. If* $n \le |\pi(G)|$, *then G is strongly supersoluble and G induces on any its non-Frattini chief factor* H/K *an automorphism group of order dividing* $p_1 \cdots p_m$, *where* $m \le n$ *and* $p_1, \dots, p_m$ *are distinct primes*.

The example of the alternating group A_4 of degree 4 shows that the restrictions on $|\pi(G)|$ in Theorem 1.12 cannot be weakened.

***Corollary* 1.13** (See Theorem B in [21]). *Suppose that G is soluble and every n-maximal subgroup M of G is of the form* $M = \langle A, B\rangle$, *where A is modular and B is S-permutable in G. If* $n \le |\pi(G)|$, *then G is strongly supersoluble and G induces on any its non-Frattini chief factor* H/K *an automorphism group of order dividing* $p_1 \cdots p_m$, *where* $m \le n$ *and* $p_1, \dots, p_m$ *are distinct primes*.

We prove also the following results.

***Theorem* 1.14.** *Let E be a normal subgroup of G and let p be a Sylow p-subgroup of E such that* $(p-1, |E|) = 1$. *If either all maximal subgroups of p are weakly S-permutable in G or every cyclic subgroup of p of order p and order* 4 (*if p* = 2 *and p is non-abelian*) *are weakly S-permutable in G*, *then E is p-nilpotent and* $E/O_{p'}(E)$ *is hypercyclically embedded in G*.

***Theorem* 1.15.** *Let E be a normal subgroup of G. If every cyclic subgroup of E of prime odd order is weakly S-permutable in G*, *then* $E/O_2(E)$ *is hypercyclically embedded in G*.

***Theorem* 1.16.** *Let E be a normal subgroup of G. Suppose that for any non-cyclic Sylow subgroup p of E every maximal subgroup of p or every cyclic subgroup of p with prime order and order* 4 (*in the case when p is a non-abelian* 2*-group*) *are weakly S-permutable in G. Then E is hypercyclically embedded in G.*

2 Proofs of the results

***Proof of Theorem* 1.3.** (i) Suppose that this assertion is false and let G be a counterexample of minimal order.

(1) *The group* G/R *is σ-soluble for every minimal normal subgroup* R *of G. Hence* R *is the unique minimal normal subgroup of G and* R *is not σ-primary*. Assume that this is false. Then G/R is not nilpotent, so G/R has a Schmidt subgroup H/R. Then H/R is soluble by Lemma 2.12 in [15], so $H < G$. Moreover, from Lemma 2.12 in [15] it follows that for every prime p dividing $|H/R|$ and for every Sylow *p*-subgroup *p* of H/R it follows that p is contained in some 2-maximal subgroup of G/R. Hence R is contained in some 3-maximal subgroup of *G*. Now let

$$M_3/R < M_2/R < M_1/R < M_0/R = G/R$$

be any maximal of G of length 3. Then $M_3 < M_2 < M_1 < M_0 = G$ is a maximal chain in G of length 3 and so for some i the subgroup M_i nearly σ-subnormal in G by hypothesis. But then M_i/R is nearly σ-subnormal in G/R by Lemma 2.11 (1) in [15]. Therefore the hypothesis holds for G/R, so the choice of G implies that G/R is σ-soluble. Hence the choice of *G* implies that R is the unique minimal normal subgroup of *G* and R is not σ-primary. Hence Claim (1) holds.

From Claim (1) it follows that R is not abelian. Let *p* be any odd prime dividing $|R|$ and R_p a Sylow *p*-subgroup of *R*. Let G_p be a Sylow *p*-subgroup of G such that $R_p = G_p \cap R$. Then $G_p \le N_G(R_p)$. Moreover, the Frattini argument implies that $G = RN_G(R_p)$. Hence there is a maximal subgroup M of *G* such that $G_p \le N_G(R_p) \le M$ and $G = RM$. Then $M \ne M_G = 1$ by Claim (1).

(2) *The subgroup M is not nearly σ-subnormal in G*. Indeed, suppose that $M = \langle A, B\rangle$, where A is some nearly modular subgroup of G and B is a σ-subnormal subgroup of G. Suppose that $A = 1$, that is, $M = B$ is a σ-subnormal subgroup of G. Then there is a subgroup chain

$$M = M_0 \le M_1 \le \dots \le M_r = G$$

such that either $M_{i-1} \trianglelefteq M_i$ or $M_i/(M_{i-1})_{M_i}$ is σ-primary for all $i = 1, \dots, r$. But M is a maximal subgroup of G and so, in fact, $M = M_{r-1}$ is not normal in *G*. Hence $G \simeq G/M_G = G/1$ is σ-primary, so G is σ-soluble. This contradiction shows that $A \neq 1$. On the other hand, $A_G \leq M_G = 1$ and $A^G / A_G \leq Z_{n\infty}(G / A_G)$. Hence $R \leq A^G \leq Z_{n\infty}(G)$ by Claim (1). But then R is abelian since every chief factor of G below $Z_{n\infty}(G)$ is soluble by Lemma 2.7 in [15]. This contradiction completes the proof of the claim.

(3) *The subgroup* $D = M \cap R$ *is not nilpotent. Hence* $D \nleq \Phi(M)$ *and* $|D|$ *is not a prime power.* Assume that D is nilpotent. Note that

$$R_p = G_p \cap R \leq M \cap R = D,$$

so R_p a Sylow p-subgroup of D. Then R_p is characteristic in D and so it is normal in M. Hence $Z(J(R_p))$ is normal in M. Since $M_G = 1$, it follows that $N_G(Z(J(R_p))) = M$ and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that R is p-nilpotent by Glauberman-Thompson's theorem on the normal p-complements [22, Chapter 8, Theorem 3.1]. But then R is a p-group, a contradiction. Hence we have (3).

(4) $R < G$. Indeed, suppose that $R = G$ is a simple non-abelian group. Let p be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$, and let L be a maximal subgroup of G containing p. Then, in view of [23, IV, Satz 2.8], $|P| > p$. Let V be a maximal subgroup of p.

If $|V| = p$, then p is abelian, so $1 < V < P < L$ by [23, IV, Satz 7.4]. On the other hand, in the case when $|V| > p$ we have $1 < W < V < P < G$, where W is a maximal subgroup of V. Hence there is a 3-maximal subgroup E of G such that $E \neq 1$. But then some proper non-identity subgroup H of G is nearly σ-subnormal in G by hypothesis. Hence $H = \langle A, B \rangle$ for some nearly modular subgroup A and some σ-subnormal subgroup B of G. Assume that $A \neq 1$. Then $A_G = 1$ and $A^G = G \leq Z_{n\infty}(G)$. Therefore G is soluble, a contradiction.

Therefore $A = 1$, so $H = B$ is σ-subnormal in G. Then there is a subgroup chain

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that either $H_{i-1} \trianglelefteq H_i$ or $H_i / (H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$. Without loss of generality, we can assume that $M = A_{n-1} < G$. Then $M_G = 1$ since $G = R$ is simple, so $G \simeq G/1$ is σ-primary. This contradiction shows that we have (4).

(5) M *is* σ*-soluble*. If the identity subgroup 1 of M is either maximal or 2-maximal in M, it is clear. Now let $L < T < M$, where L is a maximal subgroup of T and T is a maximal subgroup of M. Since M is not nearly σ-subnormal in G by Claim (2), either L or T is nearly σ-subnormal in G and so it is nearly σ-subnormal in M by Lemma 2.11 (2) in [15]. Hence the hypothesis holds for M, so M is σ-soluble by the choice of G.

(6) $M = D \rtimes T$, *where* T *is a maximal subgroup of* M *of prime order*. In view of Claim (3), there is a maximal subgroup T of M such that $M = DT$. Then $G = RM = R(DT) = RT$ and so, in view of (4), $T \neq 1$. Assume that $|T|$ is not a prime and let V be a maximal subgroup of T. Then $V \neq 1$. Since M is not nearly σ-subnormal in G, at least one of the subgroups T or V is nearly σ-subnormal in G by hypothesis. Claim (5) implies that both subgroups V and T are σ-soluble. Consider, for example, the case when V is nearly σ-subnormal in G, that is, $V = \langle A, B \rangle$ for some nearly modular subgroup A and some σ-subnormal subgroup B of G. Note that B is also σ-soluble, so in the case when $B \neq 1$ we get that $O_{\sigma_i}(V) \neq 1$ for some i. But $O_{\sigma_i}(B) \leq O_{\sigma_i}(G)$ by Lemma 2.10 (5) in [15], so $O_{\sigma_i}(G) \neq 1$, which implies that R is σ-primary by Claim (1), a contradiction.

Therefore $B = 1$, that is, $V = A$ is nearly modular in G. It is clear that $A_G = 1$ and hence $1 < A^G = V^G \leq Z_{n\infty}(G)$, which implies that $R \leq Z_{n\infty}(G)$. But then R is abelian, a contradiction. Hence $|T|$ is a prime, so $M = D \rtimes T$.

Final contradiction for (i). Since T is a maximal subgroup of M and it is cyclic, M is soluble by [23, IV, Theorem 7.4] and so $|D|$ is a prime power, contrary to Claim (3). Hence Assertion (i) is true.

(ii) Suppose that this assertion is false and let G be a counterexample of minimal order. Then G is nether σ-nilpotent nor supersoluble but every maximal subgroup M of G is σ-nilpotent. Indeed, if T is a maximal subgroup of M, then T is σ-permutable in G, so T is σ-subnormal in G by [1, Theorem B]. Hence T is σ-subnormal in M by Lemma 2.10 (1) in [15]. Hence every maximal subgroup of M is T is σ-subnormal in M, so M is σ-nilpotent by Proposition 2.3 in [1].

Therefore G is an $\mathfrak{N}_\sigma$-critical group, so G is a Schmidt group by Lemma 2.13 in [15]. Hence, in view of Lemma 2.12 in [15], $G = P \rtimes Q$, where $P = G^{\mathfrak{N}}$ is a Sylow p-subgroup of G and $Q = \langle x \rangle$ is a cyclic Sylow q-subgroup of G. Moreover, $\Phi(P)\langle x^q \rangle \leq \Phi(G)$, p is of exponent p or exponent 4 if p is a non-abelian 2-group and $P/\Phi(P)$ is a non-central chief factor of G. It is clear that $M = \Phi(P)Q$ is a maximal subgroup of G and p is the Hall σ_i-subgroup of G and q is a Hall σ_j-subgroup of G for some $p \in \sigma_i$ and $q \in \sigma_j$. Moreover, $Q^G = G$ since $P = G^{\mathfrak{N}}$.

First assume that $\Phi(P) \neq 1$ and let V be a maximal subgroup of M containing q. Then V is 2-maximal in G, so it is σ-permutable in G.

It follows that $G = Q^G \le V < G$, a contradiction. Hence $\Phi(P) = 1$, so *p* is a minimal normal subgroup of *G* and Hence *q* is a maximal subgroup of *G*. Since *G* is not supersoluble for a maximal subgroup *W* of *p* we have $W \ne 1$ and $W\langle x^q \rangle$ is a 2-maximal subgroup of *G*. Hence

$$W\langle x^q \rangle Q = QW\langle x^q \rangle,$$

which implies that *p* is not a minimal normal subgroup of *G*. This contradiction completes the proof of Statement (ii).

(iii) First note that *G* is soluble by Part (i). Therefore, in view of Proposition 2.18 in [15], we need only to show that for every maximal subgroup M of *G* we have $G / M_G \in \mathfrak{N}_n$.

If $M_G \ne 1$, then the choice of *G* and Lemma 2.15 (2) in [15] imply that $G / M_G \in \mathfrak{N}_n$. Now assume that $M_G = 1$, so there is a minimal normal subgroup R of G such that $G = R \rtimes M$ and $R = C_G(R) = O_p(G)$ for some prime *p* by [24, Chapter A, Theorem 15.6]. Lemma 2.16 (1) in [15] implies that $|M| = q$ for some prime *q* and hence R is a maximal subgroup of *G*. Then, by Lemma 2.16 (2) in [15], $|R| = p$, which implies that $|G| = pq$. Hence $G \simeq G / M_G$ is nearly nilpotent, so the Statement (iii) holds. □

***Proof of Theorem* 1.9.** *Necessity*. First suppose that *G* is not nearly σ -nilpotent and every maximal chain of length 2 in *G* includes a proper nearly σ -subnormal subgroup of *G*. We show that in this case *G* is a Schmidt group with abelian Sylow subgroups. First note that, by Proposition 4.2 in [15], for some maximal subgroup M of G we have $G / M_G \notin \mathfrak{N}_{n\sigma}$, so M is not nearly σ -subnormal in *G* by Lemma 2.11 (5) in [15]. Then every maximal subgroup V of M is nearly σ -subnormal in *G* by hypothesis. Therefore, if *M* has two different maximal subgroups V and *W*, then $M = \langle V, W \rangle$ is nearly σ -subnormal in *G* by Lemma 2.11 (4) in [15], so M possesses the unique maximal subgroup and hence *M* is a cyclic Sylow *q*-subgroup of *G* for some prime *q*. In view [23, IV, Satz 7.4], *G* is soluble.

Suppose that $M_G \ne 1$ and let R be a minimal normal subgroup of *G* contained in M_G. Then $R \le Z(G)$, since $M \le C_G(R)$ and M is a maximal subgroup of *G* which is clearly not normal in *G*. In view of Lemma 2.11 (1) in [15], the hypothesis holds for G / R, so G / R is either nearly σ -nilpotent or a Schmidt group with abelian Sylow subgroups. In the former case we have

$$G / M_G \simeq (G / R) / (M_G / R) \in \mathfrak{N}_n$$

by Proposition 4.2. But then M is nearly σ -subnormal in *G* by Lemma 2.11 (6) in [15], contrary to our assumption on *M*, and so we have the second case. Therefore, if V / R is any maximal subgroup of G / R, then V / R is nilpotent and hence V is nilpotent since $R \le Z(G)$. Therefore every maximal subgroup of *G* containing R is nilpotent. Note also that if for some maximal subgroup V of *G* we have $G = RV$, then $M = R(M \cap V)$ and so $M \cap V = 1$ since M is cyclic *q*-group. But then $M = R$ is normal in *G*, a contradiction. Therefore $R \le \Phi(G)$. Thus *G* is a Schmidt group. From Lemma 2.12 in [15] it follows that $G / R = (PR / R) \rtimes (M / R)$, where: $P \simeq PR / R = (G / R)G^{\mathfrak{N}}$ is a Sylow *p*-subgroup of G / R and *p* is a Sylow *o*-subgroup of *G*; M / R is a Sylow *q*-subgroup of G / R. Therefore all Sylow subgroups of *G* are abelian.

Now assume that $M_G = V_G = 1$. Then $G = R \rtimes M$, where $R = C_G(R)$ is a minimal normal subgroup of *G* [24, Chapter A, Theorem 15.2]. It is clear also that $R = G^{\mathfrak{N}}$. By hypothesis, $V = AB$ for some nearly modular subgroup *A* and σ -subnormal subgroup B of *G*. But since M is a cyclic Sylow subgroup of *G*, then we have that either $V = A$ or $V = B$.

First assume that $V = A$. Then

$$RV = V^G \le Z_{n\infty}(G),$$

which implies that $R / 1$ is nearly central in *G* by Lemma 2.7 in [15]. Hence

$$|G| = |R||G / C_G(R)| = |R||G / R|$$

divides *pq* for some primes *p* and *q*. Therefore *G* is either nearly σ -nilpotent or a Schmidt group with abelian Sylow subgroups.

Now consider the case when $V = B$ is σ -subnormal in *G*. Since *G* is not nearly σ -nilpotent, it is not σ -primary. Hence, in fact, $V = B$ is subnormal in *G*, then

$$V^G = V^{RM} = V^M \le M_G = 1$$

by [24, Chapter A, Theorem 14.3] and so $|M| = q$, which implies that *G* is a Schmidt group with abelian Sylow subgroups, contrary to our assumption on *G*. This contradiction completes the proof of the necessity of the condition of the theorem.

Sufficiency. If *G* is nearly σ -nilpotent, then every maximal subgroup of *G* is nearly σ -subnormal in *G* by Lemma 2.11 (6) in [15]. Finally, if *G* is a Schmidt group with abelian Sylow subgroup, then $G = R \rtimes M$, where R is a minimal normal subgroup of *G* and M_G is the maximal subgroup of M by Lemma 2.12 in [15]. Hence every 2-maximal subgroup of *G* is subnormal and so nearly σ -subnormal in *G*. □

***Proof of Theorem* 1.12.** Assume this theorem is false and let *G* be a counter example of minimal order.

I. First we show that *G* is strongly supersoluble. Suppose that this is false. Let *R* be a minimal normal subgroup of *G*.

(1) *G/R is strongly supersoluble. Hence G is primitive and so $R \nleq \Phi(G)$ and*

$$R = C_G(R) = O_p(G)$$

for some prime p. Lemma 2.15 (2) in [15] implies that the hypothesis holds for G/R, so the choice of G implies that G/R is strongly supersoluble. Therefore, again by the choice of G, R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Theorem A in [25]. Hence G is primitive and so

$$R = C_G(R) = O_p(G)$$

for some prime p by [24, Chapter A, 15.6].

(2) *Every maximal subgroup M of G is strongly supersoluble*. By hypothesis every $(n-1)$-maximal subgroup T of M is nearly S-permutable in G. Hence T is nearly S-permutable in M by Lemma 2.15 (3) in [15]. Since the solubility of G implies that either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$, the hypothesis holds for M. It follows that M is strongly supersoluble by the choice of G.

(3) *G is supersoluble*. Suppose that this is false. Since every maximal subgroup M of G is strongly supersoluble by Claim (2), G is an $\mathfrak{U}$-critical group. Then Lemma 2.19 (1) in [15] yields that $|\pi(G)| = 2$ or $|\pi(G)| = 3$. But in the former case G is strongly supersoluble by Theorem 1.5 (iii), so $|\pi(G)| = 3$ and every 3-maximal subgroup of G is nearly S-permutable in G. Claim (1) and Lemma 2.15 (3) in [15] imply that $G = R \rtimes S$, where S is a Miller – Moreno group. Moreover, since $|\pi(S)| = 2$ and S is strongly supersoluble by Claim (2), S is not nilpotent and so $S = Q \rtimes T$, where $|Q| = q$, $|T| = t$ and $C_S(Q) = Q$ for some distinct primes q and T. Hence R is a 2-maximal subgroup of G, so every maximal subgroup of R is nearly S-permutable in G. Therefore G is supersoluble by Lemma 2.16 (2) in [15].

Final contradiction for I. From Claims (1) and (3) we get that for some maximal subgroup M of G we have $G = R \rtimes M = C_G(R) \rtimes M$ and $|R| = p$, so M is cyclic. Since G is not strongly supersoluble, for some prime q dividing $|M|$ and for the Sylow q-subgroup Q of M we have $|Q| > q$. First assume that $RQ \neq G$, and let $RQ \leq V$, where V is a maximal subgroup of G. Then V is strongly supersoluble by Claim (2). Hence $C_Q(R) \neq 1$, contrary to $R = C_G(R)$. Hence $RQ = G$ and so $|\pi(G)| = 2$. Therefore G is strongly supersoluble by Theorem 1.3, a contradiction. Thus we have I.

II. Now we show that G induces on any its non-Frattini chief factor H/K an automorphism group $G/C_G(H/K)$ of order dividing $p_1 \cdots p_m$, where $m \leq n$ and $p_1, \ldots, p_m$ are distinct primes.

Let M be a maximal subgroup of G such that $K \leq M$ and $MH = G$. Then

$$G/M_G \simeq (H/K) \rtimes (G/C_G(H/K))$$

by Lemma 2.19 in [15]. If $M_G \neq 1$, the choice of G implies that $m \leq n$. Now suppose that $M_G = 1$, so $G = H \rtimes M$, where $|H|$ is a prime and $H = C_G(H)$. Then, by Claim I, M is a cyclic group of order dividing $p_1 \ldots p_m$ for some distinct primes $p_1, \ldots, p_m$. Assume that $n < m$. Then G has an n-maximal subgroup T such that $T \leq M$ and $|T|$ is not a prime, contrary to Lemma 2.16 (1) in [15]. Thus we have II. □

***Proof of Theorem* 1.14.** Suppose that this theorem is false and consider a counter example (G, E) for which $|G| + |E|$ is minimal. Let $Z = Z_{\mathfrak{n}\infty}(G)$.

(1) *If R is a normal p'-subgroup of G, then the hypothesis hold for* $(G/R, ER/R)$. First note that $PR/R \simeq P$ is a Sylow p-subgroup of ER/R, and if V/R is a subgroup of PR/R, then for a Sylow p-subgroup W of V we have $V/R = WR/R$. Moreover, if V/R is a maximal subgroup of PR/R, then $|P:W| = p$ and so W is a maximal subgroup of p. On the other hand, if V/R is a cyclic subgroup of PR/R of order p or order 4, then W is a cyclic subgroup of p of order p (respectively of order 4) since $W \simeq V/R$. Hence the hypothesis folds for $(G/R, ER/R)$ by Lemma 2.2 (1) in [26].

(2) $O_{p'}(G) = 1$. Assume that $O_{p'}(G) \neq 1$, and let R be a minimal normal subgroup of G contained in $O_{p'}(G)$. Then the hypothesis holds for $(G/R, ER/R)$ by Claim (1), so

$$(ER/R)/O_{p'}(ER/R)$$

is hypercyclically embedded in G/R and

$$ER/R \simeq E/(E \cap R)$$

is p-nilpotent. Hence E is p-nilpotent and from

$$(ER/R)/O_{p'}(ER/R) =$$
$$= (ER/R)/(O_{p'}(ER)/R) =$$
$$= (ER/R)/(O_{p'}(E)R/R)$$

and from the G-isomorphisms

$$(ER/R)/(O_{p'}(E)R/R) \simeq ER/O_{p'}(E)R \simeq$$
$$\simeq E/(E \cap O_{p'}(E)R) =$$
$$= E/O_{p'}(E)(E \cap R) = E/O_{p'}(E)$$

we get that $E/O_{p'}(E)$ is hypercyclically embedded in G, contrary to the choice of (G, E). Hence we have (2).

(3) $Z \cap E \leq Z_\infty(E)$. Indeed, since Z is clearly supersoluble, a Sylow q-subgroup q of Z, where q is the largest prime dividing $|Z|$, is normal and so characteristic in Z. Then q is normal in G, which implies that $Z = Q$ by Claim (2), so $Z \cap E \leq Z_\infty(E)$ since $(p-1, |E|) = 1$.

(4) E is p-nilpotent. Assume that this is false.

(a) $E = G$. Since the hypothesis holds for (E, E) by Lemma 2.2 (2) in [26], in the case when $E \neq G$, the subgroup E is p-nilpotent by the choice of G.

(b) *Every cyclic subgroup of p of order p and order* 4 *are m-S-supplemented in G*. It is enough to show that some maximal subgroup of p is not m-S-supplemented in G. Suppose that this is false.

First we show that $O_p(G) \neq 1$. Assume that $O_p(G) = 1$. Let V be a maximal subgroup of p. There are a nearly S-permutable subgroup S and a subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Let A be a nearly modular subgroup and B an S-permutable subgroup of G such that $S = \langle A, B \rangle$. Then $BP^x = P^x B = P^x$ for all $x \in G$, so $B \leq P_G = O_p(G) = 1$. Hence $S = A$ and $A_G = 1$, therefore $S \leq Z \leq Z_\infty(G)$ by Lemma 2.4 in [26] and Claim (3) since $E = G$ by Claim (a). Since $Z_\infty(G)$ is nilpotent, a Sylow p-subgroup of $Z_\infty(G)$ is normal in G, so $A = S = 1$ since $V_G = 1$. Therefore T is a complement to V in G, so for a Sylow p-subgroup T_p of T we have $|T_p| = p$. Therefore T is p-nilpotent since $(p-1, |E|) = (p-1, |G|) = 1$. Hence every maximal subgroup V of p has a p-nilpotent complement in G, so G is p-nilpotent by Lemma 2.13 in [26]. This contradiction shows that $O_p(G) \neq 1$.

Let R be a minimal normal subgroup of G contained in $O_p(G)$. First we show that $R \neq P$. Assume that $R = P$ and let V be any maximal subgroup of R. There are a nearly S-repmutable subgroup S and a subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Let A be a nearly modular subgroup and B an S-permutable subgroup of G such that $S = \langle A, B \rangle$. Then $A_G = 1$, so $A^G \leq Z$ by Lemma 2.4 in [26]. Therefore $A = 1$ and so $S = B$ is S-permutable in G. But then S is normal in G by Lemma 1.2.16 in [27]. Hence $S = 1$ and so $T \cap V = 1$. But then $1 < T \cap R < R$, where $T \cap R$ is normal in G. This contradiction shows that $R \neq P$. Therefore the hypothesis holds for G/R, so G/R is p-nilpotent. Hence G is p-soluble. Therefore every minimal normal subgroup R of G is a p-group by Claim (2), hence R is a unique minimal normal subgroup of G and $R \nleq \Phi(G)$, so $R = C_G(R) = O_p(G)$ by [24, Chapter A, Theorem 15.6]. It is clear also that $|R| > p$, so $Z = 1$.

Let V be a maximal subgroup of p such that $RV = P$. Then $W = V \cap R$ is normal in p, $|P : W| = p$ and $V_G = 1$. There are a nearly S-permutable subgroup S and a subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Arguing as above, we can show that S is S-permutable in G. It follows that $S \leq O_p(G) = R$. Hence $S \leq R \cap V = W$ and so $S^G = S^{PO^p(G)} = S^W \leq W$ by [27, Lemma 1.2.16], which implies that $S = 1$. Then T is a complement to V in G, so T is p-nilpotent.

Now let V be any maximal subgroup of p containing R, and let M be a maximal subgroup of G such that $G = R \rtimes M$. Then $M \simeq G/R$ is p-nilpotent, so M is a p-nilpotent supplement to V in G. Thus every maximal subgroup of p has a p-nilpotent supplement in G. Therefor G is p-nilpotent by Lemma 2.13. This contradiction shows that some maximal subgroup of p is not weakly S-permutable in G, so we have (b) by hypothesis.

Final contradiction for (4). Since G is not p-nilpotent, it has a non-nilpotent subgroup H such that every proper subgroup of H is nilpotent by [23, IV, Satz 5.4]. Moreover, Proposition 1.9 in [29, Chapter 1] implies that $H = H_p \rtimes H_q$, where H_p is a Sylow p-subgroup and H_q is a Sylow q-subgroup of H for some prime $q \neq p$ and the following hold: (i) H_p is of exponent p or exponent 4 (if $p = 2$ and H_p is non-abelian); (ii) H_p is the smallest normal subgroup of H with nilpotent quotient H/H_p; (iii) $H_p/\Phi(H_p)$ is a chief factor of H. Then $C_H(H_p/\Phi(H_p)) \neq H$ since $H/\Phi(H_p)$ is not nilpotent. Therefore $|H_p/\Phi(H_p)| > p$ since clearly $(p-1, |H|) = 1$. Lemma 2.2 (2) in [26] and Claim (b) imply that every cyclic subgroup of H_p of order p and order 4 are weakly S-permutable in H. On the other hand, Property (i) implies that $\Omega(H_p) = H_p$ and so H_p is hypercyclically embedded in H by Lemma 2.12 in [26] and so $|H_p/\Phi(H_p)| = p$. This contradiction completes the proof of (4).

The final contradiction. Claims (2) and (4) imply that $E = P$ is a normal p-subgroup of G. Let $D = \Omega(C)$, where C is a Thompson critical subgroup of E. If $D < E$, then D is hypercyclically embedded in G by the choice of (G, E) and so in this case E is hypercyclically embedded in G by Lemma 2.11 in [26]. Therefore $D = E$ and so E is hypercyclically embedded in G by Lemma 2.12 in [26]. This final contradiction completes the proof of the result. □

***Proof of Theorem* 1.15.** Suppose that this theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal.

First we show that E is $2'$-supersoluble and so E is soluble. Assume that this is false.

(a) $E = G$. *Hence every proper subgroup of G is* $2'$*-supersoluble*. Indeed, the hypothesis holds for (E, E) by Lemma 2.2 (2) in [26], so in the case when $E \neq G$ the choice of (G, E) implies that $E/O_2(E)$

is supersoluble and hence E is $2'$-supersoluble, contrary to our assumption on E. Hence $E = G$, so we have (a) by Lemma 2.2 (2) in [26].

(b) *G is soluble*. Assume that this is false and let $F = F(G)$. Then $F = \Phi(G)$ since otherwise G is soluble by Claim (a). Moreover, G/F is a non-abelian simple group and every maximal subgroup of G/F is soluble. Therefore, in view of [28], G/F is isomorphic with some of the following groups: $PSL_2(p)$, where $p > 3$ is a prime such that $p^2 + 1 \equiv 0(5)$; $PSL_2(3^p)$, where p is an odd prime; $PSL_2(2^p)$, where p is a prime; $PSL_3(3)$; Suzuki group $Sz(2^p)$, where p is an odd prime.

Let r be the largest prime dividing $|G/F|$ and let G_r be a Sylow r-subgroup of G. Then $r > 3$ by the Burnside $p^a q^b$-theorem. Let $p \neq r$ be any odd prime dividing $|G/F|$ and let C_p be a subgroup of G of order p. We show that $C_p \leq F$. Suppose that this is false. By hypothesis, C_p is weakly S-permutable in G. First assume that C_p is nearly S-permutable in G. Then there is a proper subgroup T of G such that $C_p T = G$ and $C_p \cap T = 1$. Hence $|G:T| = p$, so T is a maximal subgroup of G. By considering the permutation representation of G/T_G on the right coset of T/T_G one can see that G/T_G is isomorphic with some subgroup of the symmetric group S_p of degree p. Since T is a maximal subgroup of G, $F = \Phi(G) \leq T$. Hence $T_G = F$, so $p = r$. This contradiction shows that for every prime $2 < p < r$ dividing $|G/F|$, every subgroup C_p of G of order p with $C_p \nleq F$ is nearly S-permutable in G. Then C_p is either nearly modular or nearly S-permutable in G. Moreover, C_p is not normal in G, so in the former case we have $(C_p)^G$ is hypercyclically embedded in G by Lemma 2.4 in [26] and hence $(C_p)^G F/F = G/F$ is soluble. This contradiction shows that C_p is S-permutable in G, so $C_p \leq O_p(G)$ and hence $G = FO_p(G)$ is soluble. This contradiction completes the proof of the fact that $C_p \leq F$.

Let P be a Sylow p-subgroup of F. Then p is characteristic in F and so it is normal in G. Let $R = (G_r)^x$ for some $x \in G$. Then $V = P \rtimes R \neq G$, so V is supersoluble by Claim (a). But then $V = P \times R$ since $r > p$, so $R \leq C_G(P)$. But then $(G_r)^G \leq C_G(P)$. Since r divides $|G/F|$, $(G_r)^G \nleq F$ and hence

$$G = (G_r)^G F = (G_r)^G \Phi(G) = (G_r)^G.$$

Therefore $P \leq Z(G)$ and $P \leq \Phi(G) = F$. Now let W be a Hall p'-subgroup of F. Then

$$PW/W \leq Z(G/W) \text{ and } PW/W \leq \Phi(G/W)$$

and so p divides $|M(G/F)|$, where $M(G/F)$ is the Schur multiplicator of G/F. But

$$\pi(|M(G/F)|) \subseteq \{2,3\}$$

(see [29, Chapter 4]). Therefore $p = 3$ since $p > 2$ and so $\pi(G/F) = \{2,3,r\}$. But from the above we also know that G/F is a minimal non-soluble group, so from [29, Chapter 4] we deduce that the Schur multiplicator of G/F is of order 2. This contradiction completes the proof of the fact that G is soluble.

Let p be any odd prime. We show that G is p-supersoluble. Suppose that this is false. Then p divides $|G|$ and G is a minimal non-p-supersoluble group.

It is well-known that the class of all p-supersoluble groups $\mathfrak{F}$ is a saturated formation. Hence G has a normal subgroup D such that G/D is p-supersoluble and D is a q-group for some prime q. It is clear that $q = p$, so D is hypercyclically embedded in G by Theorem 1.14. Hence G is p-supersoluble. This contradiction completes the proof of the fact that E is $2'$-supersoluble.

Let $\{p_1, \ldots p_n\}$ be the set of all odd primes dividing $|E|$. Then $O_{p_1'}(E) \cap \cdots \cap O_{p_n'}(E) = O_2(E)$. On the other hand, $E/O_{p_i'}(E)$ is supersoluble by Theorem 1.14 for all $i = 1, \ldots, n$, so $E/O_2(E)$ is supersoluble. Assume that $O_2(E) \neq 1$. The subgroup $O_2(E)$ is characteristic in E, so it is normal in G. The hypothesis holds for $(G/O_2(E), E/O_2(E))$ by Lemma 2.2 (1) in [26], so the choice of (G,E) implies that $H/O_2(E)$ is hypercyclically embedded in $G/O_2(E)$, so $E/O_2(E)$ is hypercyclically embedded in G. Therefore $O_2(E) = 1$, so E is supersoluble.

Now let E_q be a Sylow q-subgroup of E, where q is the largest prime dividing $|E|$. Then E_q is characteristic in E, so E_q is normal G. The hypothesis holds for (G, E_q) and $(G/E_q, E/E_q)$ by Lemma 2.2 (1) in [26], so in the case when $E_q < E$, E_q and E/E_q are hypercyclically embedded in G. But then E is hypercyclically embedded in G, contrary to the choice of (G,E). Hence $E_q = E$, so E is hypercyclically embedded in G by Theorem 1.14. This final contradiction completes the proof of the result.

***Proof of Theorem* 1.16.** Suppose that this theorem is false and consider a counterexample (G,E) for which $|G| + |E|$ is minimal. Let p be the least prime dividing $|E|$ and let p be a Sylow p-subgroup of E.

Note that the hypothesis holds for (E, E) by Lemma 2.2 (2) in [26], so E is p-supersoluble by Theorem 1.14 and hence E is p-nilpotent since p is the least prime dividing $|E|$. Note also that if X is a non-identity Hall subgroup of E, then $X = E$. Indeed, the hypothesis holds for $(G/X, E/X)$ and for (G, X) by Lemma 2.2 (1) in [26]. Hence in the case $X \neq E$ the choice of G implies that E/X and X are hypercyclically embedded in G. Hence E is hypercyclically embedded in G by the Jordan-Hölder theorem for the chief series. This contradiction shows that $E = P$, so E is hypercyclically embedded in G by Theorem 1.14. The theorem is proved.

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