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О $\mathfrak{F}_h$ -НОРМАЛЬНЫХ ПОДГРУППАХ КОНЕЧНЫХ ГРУППЮфэнг Лиу¹, Хиухиан Фэнг², Жианхонг Хуанг³¹Шандонгский институт бизнеса и технологии, Янтай, Китай²Сюйчжоуский нормальный университет, Сюйчжоу, Китай³Китайский университет науки и технологии, Хейфэй, КитайON $\mathfrak{F}_h$ -NORMAL SUBGROUPS OF FINITE GROUPSYufeng Liu¹, Xiuxian Feng², Jianhong Huang³¹Shandong Institute of Business and Technology, Yantai, China²Xuzhou Normal University, Xuzhou, China³University of Science and Technology of China, Hefei, China

Пусть G конечная группа и $\mathfrak{F}$ – формация конечной группы. Мы говорим, что подгруппа H группы G является $\mathfrak{F}_h$ -нормальной в G если существует такая нормальная подгруппа T группы G , что HT – нормальная халловская подгруппа в G и $(H \cap T)H_G/H_G$ содержится в $\mathfrak{F}$ -гиперцентре $Z_\infty^\mathfrak{F}(G/H_G)$ группы G/H_G . В данной работе мы получаем некоторые результаты о $\mathfrak{F}_h$ -нормальных подгруппах и используем их для изучения конечных групп.

Ключевые слова: конечная группа, $\mathfrak{F}_h$ -нормальная подгруппа, подгруппа Силова, максимальная подгруппа, минимальная подгруппа.

Let G be a finite group and $\mathfrak{F}$ a formation of finite groups. We say that a subgroup H of G is $\mathfrak{F}_h$ -normal in G if there exists a normal subgroup T of G such that HT is a normal Hall subgroup of G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$ -hypercenter $Z_\infty^\mathfrak{F}(G/H_G)$ of G/H_G . In this paper, we obtain some results about the $\mathfrak{F}_h$ -normal subgroups and use them to study the structure of finite groups.

Keywords: finite groups, $\mathfrak{F}_h$ -normal subgroup, Sylow subgroup, maximal subgroup, minimal subgroup.

Introduction

Throughout this paper, all groups are finite and G denotes a group. The notation and terminology are standard, as in [1] and [2].

The relationship between the subgroups and the structure of G has been extensively studied in the literature. Many useful results of finite groups have been obtained under the assumption that some certain subgroups of G of prime power orders are well situated in G . Ito [3] has proved that if G is a group of odd order and all minimal subgroups of G lie in the center of G , then G is nilpotent. Buckley [4] showed that a group G of odd order is supersoluble if all minimal subgroups of G are normal in G . Srinivasan [5] proved that a group G is supersoluble if every maximal subgroup of every Sylow subgroup of G is normal in G .

Recently, by considering some special supplemented subgroups, people have obtained a series of new interesting results. For example, Wang [6] introduced c -normal subgroup: a subgroup H of G is said to be c -normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal

subgroup of G contained in H . Later, Yang and Guo [7] gave the concept of $\mathfrak{F}_n$ -supplemented subgroup: a subgroup H of G is said to be $\mathfrak{F}_n$ -supplemented in G if there exists a normal subgroup K of G such that $G = HK$ and $(H \cap K)H_G/H_G$ is contained in the $\mathfrak{F}$ -hypercenter $Z_\infty^\mathfrak{F}(G/H_G)$ of G/H_G . Many facts have shown that c -normal and $\mathfrak{F}_n$ -supplemented properties of some subgroups can give a good insight into the structure of supersoluble groups and p -nilpotent groups (see [6]–[12]).

As a development of this topic, the authors introduce the following new concept.

Definition 0.1 [13]. Let $\mathfrak{F}$ be a class of groups. A subgroup H of G is said to be $\mathfrak{F}_h$ -normal in G if there exists a normal subgroup T of G such that HT is a normal Hall subgroup of G and $(H \cap T)H_G/H_G \leq Z_\infty^\mathfrak{F}(G/H_G)$.

Recall that, for a class $\mathfrak{F}$ of groups, a chief factor H/K of G is called $\mathfrak{F}$ -central (see [14] or [1, Definition 2.4.3]) if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$.

The symbol $Z_{\infty}^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$ -hypercenter of a group G , that is, the product of all such normal subgroups H of G whose G -chief factors are $\mathfrak{F}$ -central. A subgroup H of G is said to be $\mathfrak{F}$ -hypercenter in G if $H \leq Z_{\infty}^{\mathfrak{F}}(G)$.

A class $\mathfrak{F}$ of groups is called a formation if it is closed under homomorphic image and every group G has a smallest normal subgroup (called $\mathfrak{F}$ -residual of G and denoted by $G^{\mathfrak{F}}$) with quotient in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. We use $\mathfrak{N}$, $\mathfrak{U}$ to denote the formations of all nilpotent groups and supersoluble groups, respectively.

Obviously, all normal subgroups, c -normal subgroups and $\mathfrak{F}_n$ -supplemented subgroups are all $\mathfrak{F}_n$ -normal in G , for any nonempty saturated formation $\mathfrak{F}$. However, the converse is not true in general (see [13, Example 1.2]).

In this paper, we will use $\mathfrak{F}_n$ -normal subgroups to give some new characterizations of some classes of groups. Some previously known results are generalized.

1 Preliminaries

A formation $\mathfrak{F}$ is said to be S -closed (S_n -closed) if it contains all subgroups (all normal subgroups, respectively) of all its groups.

For the sake of convenience, we cite the following lemmas which are useful in this paper.

Lemma 1.1 [15, Lemma 2.1]. *Let $\mathfrak{F}$ be a non-empty saturated formation, $A \leq G$ and $Z = Z_{\infty}^{\mathfrak{F}}(G)$. Then*

- (1) *If A is normal in G , then $AZ/A \leq Z_{\infty}^{\mathfrak{F}}(G/A)$.*
- (2) *If $\mathfrak{F}$ is S -closed, then $Z \cap A \leq Z_{\infty}^{\mathfrak{F}}(A)$.*
- (3) *If $\mathfrak{F}$ is S_n -closed and A is normal in G , then $Z \cap A \leq Z_{\infty}^{\mathfrak{F}}(A)$.*
- (4) *If $G \in \mathfrak{F}$, then $Z = G$.*

Lemma 1.2 [12, Lemma 2.5]. *Let p be a prime number such that $(|G|, p^2 - 1) = 1$. If G/L is p -nilpotent and $p^3 \nmid |L|$, then G is p -nilpotent.*

Lemma 1.3 [16, II 7.9]. *Let N be a nilpotent normal subgroup of G . If $N \neq 1$ and $N \cap \Phi(G) = 1$, then N is a direct product of some minimal normal subgroups of G .*

Lemma 1.4 [3, VI 14.3]. *Let G be a finite group. If G has an abelian Sylow p -subgroup P of G , then $Z(G) \cap G' \cap P = 1$.*

Lemma 1.5 [17, Theorem 1]. *Let $\mathfrak{F}$ be a saturated formation and G be a minimal non- $\mathfrak{F}$ -group such that $(G^{\mathfrak{F}})'$ is a proper subgroup of $G^{\mathfrak{F}}$, then $G^{\mathfrak{F}}$ is a solvable group.*

Lemma 1.6 [1, Corollary 3.2.9]. *If $\mathfrak{F}$ is a local formation, then $[G^{\mathfrak{F}}, Z_{\infty}^{\mathfrak{F}}(G)] = 1$, for any group G .*

Lemma 1.7 [13, Theorem 3.2]. *Let $\mathfrak{F}$ be a S -closed saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If all cyclic subgroups of E of prime order and order 4 are $\mathfrak{U}_n$ -normal in G , then $G \in \mathfrak{F}$.*

Lemma 1.8 [13, Theorem 4.1]. *A group G is solvable if and only if every minimal subgroup of G is $\mathfrak{U}_n$ -normal in G .*

Lemma 1.9 [13, Theorem 3.1]. *A group G is supersoluble if and only if there exists a normal subgroup E of G such that G/E is supersoluble and every maximal subgroup of every non-cyclic Sylow subgroup of E is $\mathfrak{U}_n$ -normal in G .*

Lemma 1.10. *Let R be a soluble minimal normal subgroup of G . If there exists a maximal subgroup R_1 of R such that R_1 is $\mathfrak{U}_n$ -normal in G , then R is a group of prime order.*

Proof. Since R is a minimal normal subgroup of G , R is an elementary abelian group and $(R_1)_G = 1$. By hypothesis, there exists a normal subgroup K of G such that R_1K is a normal Hall subgroup of G and $R_1 \cap K \leq Z_{\infty}^{\mathfrak{U}}(G)$. Since $R \cap KG$, $R \cap K = 1$ or $R \cap K = R$. If $R \cap K = 1$, then $R = R \cap R_1K = R_1(R \cap K) = R_1$, a contradiction. If $R \cap K = R$, then $R \leq K$, and so $R_1 \leq K$. It follows that $R_1 = R_1 \cap K \leq Z_{\infty}^{\mathfrak{U}}(G)$. If R is not a group of prime order, then $1 \neq R_1 \leq Z_{\infty}^{\mathfrak{U}}(G) \cap R$. Hence $Z_{\infty}^{\mathfrak{U}}(G) \cap R \neq 1$ and $R \leq Z_{\infty}^{\mathfrak{U}}(G)$. It follows that R is a group of prime order. This contradiction completes the proof.

Lemma 1.11 [13, Lemma 2.6]. *Let G be a group and $H \leq K \leq G$. Then*

- (1) *H is $\mathfrak{F}_n$ -normal in G if and only if G has a normal subgroup T such that HT is a normal Hall subgroup of G , $H_G \leq T$ and $H/H_G \cap T/H_G \leq Z_{\infty}^{\mathfrak{F}}(G/H_G)$.*
- (2) *Suppose that H is normal in G . If K is $\mathfrak{F}_n$ -normal in G , then K/H is $\mathfrak{F}_n$ -normal in G/H .*
- (3) *Suppose that H is normal in G . Then for every $\mathfrak{F}_n$ -normal subgroup E of G satisfying $(|H|, |E|) = 1$, HE/H is $\mathfrak{F}_n$ -normal in G/H .*
- (4) *If H is $\mathfrak{F}_n$ -normal in G and $\mathfrak{F}$ is S -closed, then H is $\mathfrak{F}_n$ -normal in K .*

(5) *If H is $\mathfrak{F}_n$ -normal in G and $\mathfrak{F}$ is S_n -closed, then H is $\mathfrak{F}_n$ -normal in K .*

(6) *If $G \in \mathfrak{F}$, then every subgroup of G is $\mathfrak{F}_n$ -normal in G .*

2 Main Results

Theorem 2.1. *Let p be a prime divisor of $|G|$ with $(|G|, p-1)=1$. Then G is p -nilpotent if and only if there exists a normal subgroup N of G such that G/N is p -nilpotent and every maximal subgroup of every Sylow subgroup of N is $\mathfrak{U}_h$ -normal in G .*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. We proceed the proof via the following steps.

(1) G is soluble, G has a minimal normal subgroup $L \leq N$ and L is an elementary abelian r -group, where r is the largest prime in $\pi(N)$.

If $p > 2$, then G is soluble since $(|G|, p-1)=1$. Now we assume that $p=2$. Then G/N is 2-nilpotent and so G/N is soluble. Since every maximal subgroup of every Sylow subgroup of N is $\mathfrak{U}_h$ -normal in G , it is $\mathfrak{U}_h$ -normal in N by Lemma 1.11 (4). Applying Lemma 1.9 for the case $G=N$, we get that N is supersoluble and so G is soluble. Hence, for the largest prime number r in $\pi(N)$, the Sylow r -subgroup R of N is normal in N . Since $R \text{ char } N \trianglelefteq G$, R is normal in G . Thus, G has a minimal normal subgroup $L \leq N$ and L is an elementary abelian r -group.

(2) G/L is p -nilpotent and $L = R \in \text{Syl}_p(N)$.

Obviously, $(G/L)/(N/L) \cong G/N$ is p -nilpotent. Let R_1/L be a maximal subgroup of a Sylow r -subgroup of N/L . Then R_1 is a maximal subgroup of the Sylow r -subgroup R of N . By hypothesis and Lemma 1.11 (2), R_1/L is $\mathfrak{U}_h$ -normal in G/L . Let Q_1/L be a maximal subgroup of a Sylow q -subgroup of N/L , where $q \neq r$. It is clear that $Q_1 = Q_1^*L$, where Q_1^* is a maximal subgroup of a Sylow q -subgroup of N . By hypothesis and Lemma 1.11 (3), $Q_1/L = Q_1^*L/L$ is $\mathfrak{U}_h$ -normal in G/L . Hence by the minimal choice of G , G/L is p -nilpotent. If $p \nmid |L|$, then G is p -nilpotent, a contradiction. So L is a p -group. Since the class of all p -nilpotent groups is a saturated formation, L is the unique minimal normal subgroup of G contained in N and $L \not\leq \Phi(G)$. By Lemma 1.3, $F(N)=L$. Since N is soluble, $L \leq C_N(F(N)) \leq F(N)$ and so $C_N(L) = L = F(N)$. Because RG and $R \leq F(N)$, we have that $L = R \in \text{Syl}_p(N)$.

(3) Final contradiction.

Let L_1 be a maximal subgroup of L . By (2) and the hypothesis, L_1 is $\mathfrak{U}_h$ -normal in G . Then by

Lemma 1.10, we have that $|L|=p$. Since G/L is p -nilpotent, G/L has a normal p -complement H/L . By Schur Zassenhaus theorem, $H = G_p L$, where G_p is a Hall p' -subgroup of G . Since p is the prime divisor of $|G|$ with $(|G|, p-1)=1$ and $N_H(L)/C_H(L) \lesssim \text{Aut}(L)$ is a cyclic subgroup of order $p-1$. By the well known Burnside theorem, we have that H is p -nilpotent. Hence, $G_p \text{ char } H \trianglelefteq G$ and so $G_p \trianglelefteq G$. Clearly, G_p is a normal p -complement of G , which implies that G is p -nilpotent. The final contradiction completes the proof.

Theorem 2.2. *Let p be a prime divisor of $|G|$ with $(|G|, p-1)=1$. Then G is p -nilpotent if and only if G has a soluble normal subgroup H such that G/H is p -nilpotent and every maximal subgroup of every Sylow subgroup of $F(H)$ is $\mathfrak{U}_h$ -normal in G .*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample with $|G||H|$ is minimal. Let P be an arbitrary Sylow r -subgroup of $F(H)$. Since $P \text{ char } F(H) \text{ char } H \trianglelefteq G$, $P \trianglelefteq G$. We proceed the proof via the following steps.

(1) $\Phi(G) \cap P = 1$.

If not, then $1 \neq \Phi(G) \cap P \trianglelefteq G$. Let $R = \Phi(G) \cap P$. Clearly, $(G/R)/(H/R) \cong G/H$ is p -nilpotent. By [3, Theorem III.3.5], we have that $F(H/R) = F(H)/R$. Assume that P/R is a Sylow r -subgroup of $F(H/R)$ and P_1/R a maximal subgroup of P/R . Then P_1 is a maximal subgroup of P . By hypothesis, P_1 is $\mathfrak{U}_h$ -normal in G . Then by Lemma 1.11 (2), P_1/R is $\mathfrak{U}_h$ -normal in G/R . Now, let Q/R be a maximal subgroup of some Sylow q -subgroup of $F(H/R) = F(H)/R$, where $q \neq r$. Then $Q = Q_1R$, where Q_1 is a maximal subgroup of the Sylow q -subgroup of $F(H)$. By hypothesis, Q_1 is $\mathfrak{U}_h$ -normal in G . Hence $Q/R = Q_1R/R$ is $\mathfrak{U}_h$ -normal in G/R by Lemma 1.11 (3). This shows that $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of (G, H) implies that G/R is p -nilpotent. Since $G/\Phi(G) \cong (G/R)/(\Phi(G)/R)$ is p -nilpotent and the class of all p -nilpotent groups is a saturated formation, G is p -nilpotent, a contradiction. Hence (1) holds.

(2) $P = \langle x_1 \rangle \times \cdots \times \langle x_m \rangle$, where every $\langle x_i \rangle$

($i = 1, \dots, m$) is a normal subgroup of G of order r .

By (1) and Lemma 1.3, $P = R_1 \times \dots \times R_m$, where R_i ($i = 1, \dots, m$) is a minimal normal subgroup of G . We now prove that all R_i are of order r .

Assume that $|R_i| > r$, for some i . Without loss of generality, we let $|R_1| > r$. Let R_1^* be a maximal subgroup of R_1 . Then $R_1^* \times R_2 \times \dots \times R_m = P_1$ is a maximal subgroup of P . Set $T = R_2 \times \dots \times R_m$, then, clearly $(P_1)_G = T$. By hypothesis, P_1 is $\mathcal{U}_h$ -normal in G . Hence by Lemma 1.11 (1), there exists a normal subgroup N of G , such that $(P_1)_G \leq N$, $P_1 N$ is a normal Hall subgroup of G and $(P_1 \cap N)/(P_1)_G \leq Z_\infty^u(G/(P_1)_G)$. It follows that $P_1 N = R_1^* T N = R_1^* N$. If $R_1^* \cap N \neq 1$, then $1 < R_1 \cap N \trianglelefteq G$. By the minimal normality of R_1 , $R_1 \cap N = R_1$ and so $R_1 \leq N$. Hence $P_1 N = R_1^* N = N$. Consequently $P_1 \leq N$. It follows that $P_1/(P_1)_G \leq Z_\infty^u(G/(P_1)_G)$. If $(P_1)_G = P_1$, then $R_1^* = 1$, which contradicts $R_1^* \cap N \neq 1$. Hence $(P_1)_G < P_1$ and so $1 \neq P_1/(P_1)_G \leq Z_\infty^u(G/(P_1)_G) \cap P/(P_1)_G$. Since $P/(P_1)_G = R_1$ and R_1 is a minimal normal subgroup of G , $P/(P_1)_G$ is a chief factor of G . This implies that $Z_\infty^u(G/(P_1)_G) \cap P/(P_1)_G = P/(P_1)_G$ and so $P/(P_1)_G \leq Z_\infty^u(G/(P_1)_G)$. It follows that $|P/(P_1)_G| = r$. Hence $|R_1| = r$, a contradiction. Now assume that $R_1^* \cap N = 1$. Then $(R_1^*)_G = 1 \leq N \trianglelefteq G$, $R_1^* N = P_1 N$ is a normal Hall subgroup and $(R_1^* \cap N)/(R_1^*)_G = 1 \leq Z_\infty^u(G/(R_1^*)_G)$. This shows that R_1^* is $\mathcal{U}_h$ -normal in G . Hence R_1 is a cyclic group of order r by Lemma 1.10, a contradiction again. Thus (2) holds.

(3) $G/F(H)$ is p -nilpotent.

From (2), $F(H) = \langle y_1 \rangle \times \dots \times \langle y_n \rangle$, where every $\langle y_i \rangle$ ($i = 1, \dots, n$) is a normal subgroup of G of prime order. Since $G/C_G(\langle y_i \rangle)$ is isomorphic to a subgroup of $\text{Aut}(\langle y_i \rangle)$, $G/C_G(\langle y_i \rangle)$ is cyclic and so it is p -nilpotent for each i . It follows that $G/\cap_{i=1}^n C_G(\langle y_i \rangle)$ is p -nilpotent. Obviously, $C_G(F(H)) = \cap_{i=1}^n C_G(\langle y_i \rangle)$. Hence $G/C_G(F(H))$ is p -nilpotent. Consequently $G/(H \cap C_G(F(H))) = G/C_H(F(H))$ is p -nilpotent. Because $F(H)$ is abelian, we have that $F(H) \leq C_H(F(H))$. On the other hand, $C_H(F(H)) \leq F(H)$ for H is soluble. Thus $F(H) = C_H(F(H))$ and so $G/F(H)$ is p -nilpotent.

(4) Final contradiction.

In view of Theorem 2.1, we have that G is p -nilpotent. The final contradiction completes the proof.

Theorem 2.3. Let $\mathfrak{F}$ be a S -closed saturated formation which satisfies that every minimal non- $\mathfrak{F}$ -group is soluble. Then G is an $\mathfrak{F}$ -group if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every cyclic subgroup of order 4 of N is $\mathfrak{F}_h$ -normal in G and every minimal subgroup of N is contained in $Z_\infty^{\mathfrak{F}}(G)$.

Proof. The necessity is obvious. We only need to prove the sufficiency. Assume that the assertion is false and choose G to be a counterexample of minimal order. Then, obviously $N \neq 1$.

Let L be a proper subgroup of G . Then $L/L \cap N \cong LN/N \leq G/N$ implies that $L/L \cap N \in \mathfrak{F}$. Since $L \cap N \leq N$, by hypothesis, every cyclic subgroup of $L \cap N$ of order 4 is $\mathfrak{F}_h$ -normal in G and hence is $\mathfrak{F}_h$ -normal in L by Lemma 1.11. On the other hand, since every minimal subgroup of L is a minimal subgroup of G , every minimal subgroup of L is contained in $Z_\infty^{\mathfrak{F}}(G) \cap L \subseteq Z_\infty^{\mathfrak{F}}(L)$ by Lemma 1.1. This shows that $(L, N \cap L)$ satisfies the hypothesis. By the minimal choice of G , $L \in \mathfrak{F}$ and so G is a minimal non- $\mathfrak{F}$ -group. By [1, Theorem 3.4.2] and the hypothesis, we know that G is soluble and G has the following properties: (1) $G^{\mathfrak{F}}$ is a p -group, for some prime p ; (2) $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G ; (3) If $G^{\mathfrak{F}}$ is abelian, then $G^{\mathfrak{F}}$ is an elementary abelian p -group; (4) If $p > 2$, then the exponent of $G^{\mathfrak{F}}$ is p ; If $p = 2$, then the exponent of $G^{\mathfrak{F}}$ is 2 or 4.

Since $G/N \in \mathfrak{F}$, $G^{\mathfrak{F}} \leq N$. Suppose that the exponent of $G^{\mathfrak{F}}$ is a prime. Then by hypothesis, $G^{\mathfrak{F}} \subseteq Z_\infty^{\mathfrak{F}}(G)$ and so $G \in \mathfrak{F}$, a contradiction.

Now assume that $G^{\mathfrak{F}}$ is not abelian and $p = 2$. We claim that there is no an element of order 4 in $G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$. Assume that there exists an element $x \in G^{\mathfrak{F}} \setminus \Phi(G^{\mathfrak{F}})$ with $|x| = 4$. Then by hypothesis, $\langle x \rangle$ is $\mathfrak{F}_h$ -normal in G . Hence by Lemma 1.11 (1), there exists a normal subgroup T of G such that $\langle x \rangle T$ is a normal Hall subgroup of G and $\langle x \rangle / \langle x \rangle_G \cap T / \langle x \rangle_G \leq Z_\infty^{\mathfrak{F}}(G / \langle x \rangle_G)$. Thus $G^{\mathfrak{F}} \leq \langle x \rangle T$ by (1). Let $P_1 = G^{\mathfrak{F}} \cap T$. Then $P_1 \trianglelefteq G$. If $P_1 \leq \Phi(G^{\mathfrak{F}})$, then $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap \langle x \rangle T = \langle x \rangle (G^{\mathfrak{F}} \cap T) = \langle x \rangle P_1 = \langle x \rangle$, a contradiction. So $P_1 \not\leq \Phi(G^{\mathfrak{F}})$. By (2) $P_1 \Phi(G^{\mathfrak{F}}) / \Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}} / \Phi(G^{\mathfrak{F}})$. It follows that

$P_1 = G^\mathfrak{F}$ and so $G^\mathfrak{F} \leq T$. Thus $\langle x \rangle \leq T$ and $\langle x \rangle = \langle x \rangle \cap T$. We first assume that $\langle x \rangle / \langle x \rangle_G \cap T / \langle x \rangle_G = 1$. Then $\langle x \rangle = \langle x \rangle_G \trianglelefteq G$. Hence $\langle x \rangle \Phi(G^\mathfrak{F}) / \Phi(G^\mathfrak{F}) \trianglelefteq G / \Phi(G^\mathfrak{F})$. Then by (2), $\langle x \rangle \Phi(G^\mathfrak{F}) = G^\mathfrak{F}$ and so $\langle x \rangle = G^\mathfrak{F}$, a contradiction. Hence $\langle x \rangle / \langle x \rangle_G \cap T / \langle x \rangle_G \neq 1$, that is $\langle x \rangle_G < \langle x \rangle$, and so $|\langle x \rangle_G| \leq 2$. If $|\langle x \rangle_G| = 1$, then $\langle x \rangle \leq Z_\infty^\mathfrak{F}(G)$. By hypothesis, $G^\mathfrak{F} \leq Z_\infty^\mathfrak{F}(G)$ and consequently $G \in \mathfrak{F}$, a contradiction. If $|\langle x \rangle_G| = 2$, then $\langle x \rangle / \langle x \rangle_G \leq Z_\infty^\mathfrak{F}(G / \langle x \rangle_G)$ and $\langle x \rangle_G \leq Z_\infty^\mathfrak{F}(G)$. It follows that $Z_\infty^\mathfrak{F}(G / \langle x \rangle_G) = Z_\infty^\mathfrak{F}(G) / \langle x \rangle_G$. Hence $\langle x \rangle \leq Z_\infty^\mathfrak{F}(G)$. This implies that $G^\mathfrak{F} \leq Z_\infty^\mathfrak{F}(G)$. Consequently $G \in \mathfrak{F}$. This final contradiction completes the proof.

Corollary 2.3.1. Let $\mathfrak{F}$ be a S -closed saturated formation which satisfies that every minimal non- $\mathfrak{F}$ -group is soluble. Then G is an $\mathfrak{F}$ -group if and only if every cyclic subgroup of order 4 of G is $\mathfrak{F}_h$ -normal in G and every minimal subgroup of G is contained in $Z_\infty^\mathfrak{F}(G)$.

Corollary 2.3.2 (Miao, Guo [18]). Let $\mathfrak{F}$ be a S -closed saturated formation which satisfies that a minimal non- $\mathfrak{F}$ -group is soluble and its $\mathfrak{F}$ -residual is a Sylow subgroup. If every cyclic subgroup of order 4 of G is c -normal in G and every minimal subgroup of G is contained in the $\mathfrak{F}$ -hypercenter of G , then G is an $\mathfrak{F}$ -group.

Corollary 2.3.3 (Miao, Guo [18]). Let $\mathfrak{F}$ be a S -closed saturated formation which satisfies that a minimal non- $\mathfrak{F}$ -group is soluble and its $\mathfrak{F}$ -residual is a Sylow subgroup. Let N be a normal subgroup of G and $G/N \in \mathfrak{F}$. If every cyclic subgroup of order 4 of N is c -normal in G and every minimal subgroup of N is contained in the $\mathfrak{F}$ -hypercenter of G , then G is an $\mathfrak{F}$ -group.

Theorem 2.4. Let $\mathfrak{F}$ be a S -closed saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup N of G such that $G/N \in \mathfrak{F}$ and all elements of N of odd prime order are $\mathfrak{U}_h$ -normal in G and N has an abelian Sylow 2-subgroup and every subgroup of N of order 2 is contained in $Z_\infty^\mathfrak{U}(G)$.

Proof. Assume that the theorem is false and let G be a counterexample of minimal order.

First we show that $M \in \mathfrak{F}$ for every maximal subgroup M of G . If $N \not\leq M$, then $G = MN$ and $M/M \cap N \cong MN/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is S -closed,

$M \cap Z_\infty^\mathfrak{F}(G) \leq Z_\infty^\mathfrak{F}(M)$ by Lemma 1.1. Then by Lemma 1.11, we see that $(M, M \cap N)$ satisfies the hypothesis. Hence $M \in \mathfrak{F}$ by the choice of G . Therefore G is a minimal non- $\mathfrak{F}$ -group. Let $R = G^\mathfrak{F}$. Then $R \leq N$. Assume that $R' < R$, where R' is the derived subgroup of R . Then R is soluble by Lemma 1.5. Hence by [1, Theorem 3.4.2] and since R has an abelian Sylow 2-subgroup, R is a p -group of exponent p . If $p \neq 2$, then $G \in \mathfrak{F}$ by Lemma 1.7, a contradiction. Suppose that $p = 2$, then R is an elementary abelian 2-group. Thus, by hypothesis, $R \leq Z_\infty^\mathfrak{F}(G)$ and so $G \in \mathfrak{F}$, a contradiction. Now assume that $R = R'$. Let T be a Sylow 2-group of R . Then T is abelian and so $T \cap Z(R) = 1$ by Lemma 1.4. Assume that $T \neq 1$. Then there exists an element $r \in T$ with $|r| = 2$. Hence $r \in Z_\infty^\mathfrak{F}(G)$ and so $r \in Z_\infty^\mathfrak{F}(G) \cap R$. Since $Z_\infty^\mathfrak{F}(G) \cap R$ is contained in $Z(R)$ by Lemma 1.6, $r \in Z(R) \cap T \neq 1$. That is $Z(R) \cap T \neq 1$. This contradiction shows that R is of odd order. Therefore by Feit-Thompson theorem, R is soluble, which contradicts $R = R'$.

These contradictions show that the counterexample of minimal order does not exist. Therefore the Theorem holds.

Theorem 2.5. Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if one of the following conditions holds:

- (a) G is 2-nilpotent and every element x of odd prime order of H is $\mathfrak{U}_h$ -normal in G .
- (b) H has an abelian Sylow 2-subgroup and every subgroup of prime order of H is $\mathfrak{U}_h$ -normal in G .

Proof. (a) If G is 2-nilpotent, then H is 2-nilpotent. Let K be the 2-complement of H . Then $K \trianglelefteq G$. Since $(G/K)/(H/K) = G/H \in \mathfrak{F}$ and H/K is a 2-group, H/K has no element of odd order. Hence $G/K \in \mathfrak{F}$ by induction on $|G|$. Since K is a 2-complement of H , K has no cyclic subgroup of order 4. Thus $G \in \mathfrak{F}$ by Lemma 1.7.

(b) Let $E = G^\mathfrak{F}$. Then, obviously, $E \leq H$ and E has abelian Sylow 2-subgroups. By hypotheses, every subgroup $\langle x \rangle$ of prime order of E is $\mathfrak{U}_h$ -normal in G . Hence, by Lemma 1.11, $\langle x \rangle$ is also $\mathfrak{U}_h$ -normal in E . It follows from Lemma 1.8 that E is soluble. Let M be a maximal subgroup of G such that $E \not\leq M$. Then $ME/E \cong M/M \cap E \in \mathfrak{F}$. It is easy to see that $(M, M \cap E)$ satisfies the hypothesis. Therefore $M \in \mathfrak{F}$ by induction. Then, applying [1,

Theorem 3.4.2], we see that E is a p -group of exponent p . Thus $G \in \mathfrak{F}$ by Lemma 1.7.

REFERENCES

1. Guo, W. The Theory of Classes of Groups / W. Guo. – Beijing-New York-Dordrecht-Boston-London : Science Press / Kluwer Academic Publishers, 2000.
2. Robinson, D.J.S. A Course in the Theory of Groups / D.J.S. Robinson. – New York : Springer, 1982.
3. Huppert, B. Endliche Gruppen I / B. Huppert. – Berlin-Heidelberg-New York : Springer-Verlag, 1967.
4. Buckley, J. Finite groups whose minimal subgroups are normal / J. Buckley // Math. Z. – 1970. – Vol. 116. – P. 15–17.
5. Srinivasan, S. Two sufficient conditions for supersolvability of finite groups / S. Srinivasan // Israel J. Math. – 1980. – Vol. 3, № 35. – P. 210–214.
6. Wang, Y. c -normality of groups and its properties / Y. Wang // J. Algebra. – 1996. – Vol. 180. – P. 954–965.
7. Yang, N. On $\mathfrak{F}_n$ -supplemented subgroups of finite groups / N. Yang, W. Guo // Asian-European Journal of Mathematics. – 2008. – Vol. 1, № 4. – P. 619–629.
8. Wang, Y. c -normality and solvability of groups / Y. Wang // J. Pure Appl. Algebra. – 1996. – Vol. 110. – P. 315–320.
9. Guo, X. On c -normal maximal and minimal subgroups of Sylow p -subgroups of finite groups / X. Guo, K.P. Shum // Arch. Math. – 2003. – Vol. 80. – P. 561–569.
10. Li, D. The influence of c -normality of subgroups on the structure of finite groups / D. Li, X. Guo // J. Pure. App. Algebra. – 2000. – Vol. 150. – P. 53–60.
11. Li, D. The influence of c -normality of subgroups on the structure of finite groups II / D. Li, X. Guo // Comm. Algebra. – 1998. – Vol. 26. – P. 1913–1922.
12. Miao, L. New criteria for p -nilpotency of finite groups / L. Miao, W. Guo, K.P. Shum // Comm. Algebra. – 2007. – Vol. 35. – P. 965–974.
13. Feng X., Guo W., Huang J., New characterizations of some classes of finite groups, Malaysian Mathematical Science Society, to appear.
14. Shemetkov, L.A. Formations of Algebraic Systems / L.A. Shemetkov, A.N. Skiba. – Moscow : Nauka, 1989.
15. Guo, W. On $\mathfrak{F}$ -supplemented subgroups of finite group / W. Guo // Manuscripta Math. – 2008. – Vol. 127. – P. 139–150.
16. Shemetkov, L.A. Formations of Finite Groups / L.A. Shemetkov. – Moscow : Nauka, 1978.
17. Ballester-Belinchés, A. On minimal subgroups of finite groups / A. Ballester-Belinchés, M.C. Pedraza-Aguilera // Acta Math. – 1966. – Vol. 73, № 4. – P. 335–342.
18. Miao, L. The influence of c -normality of some subgroups on the structure of a finite / L. Miao, W. Guo // Problems in Algebra. – 2000. – Vol. 3, № 16. – P. 101–106.

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