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S-C-ПЕРЕСТАНОВОЧНО ПОГРУЖЕННЫЕ ПОДГРУППЫ КОНЕЧНЫХ ГРУПП

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S-C-PERMUTABLY EMBEDDED SUBGROUPS OF FINITE GROUPS

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Подгруппа H конечной группы G называется s-условно перестановочно погруженной (или более кратко, s-c-перестановочно погруженной) в G если для каждого $p \in \pi(H)$, каждая силовская p-подгруппа группы H является ситловской p-подгруппой некоторой s-условно перестановочной подгруппой группы G. В данной работе мы использует некоторые s-c-перестановочно погруженные подгруппы для изучения структуры некоторых конечных групп. Обобщаются некоторые известные результаты.

Ключевые слова: *конечная группа, s-условно перестановочно погруженная подгруппа, формация, подгруппа Силова, максимальная подгруппа.*

A subgroup H of a finite group G is said to be s-conditionally permutably embedded (or in brevity, s-c-permutably embedded) in G if for each $p \in \pi(H)$, every Sylow p-subgroup of H is a Sylow p-subgroup of some s-conditionally permutable subgroup of G. In this paper, we use some s-c-permutably embedded subgroups to study the structure of some groups. Some known results are generalized.

Keywords: *finite group, s-conditionally permutably embedded subgroup, formation, Sylow subgroup, maximal subgroup.*

Introduction

Throughout this paper, all groups considered are finite and G denotes a finite group. The terminology and notations are standard, as in [1] and [2].

Let A and B be subgroups of G. A is said to be permutable with B if $AB = BA$. If A is permutable with all subgroups of G, then A is said to be a permutable subgroup [1] (or quasinormal subgroup [3]) of G. The permutable subgroups have many interesting properties. For example, Ore [3] proved that every permutable subgroup of a finite group is subnormal. Itô and Szép [4] proved that for every permutable subgroup H of a finite group G, H/H_G is nilpotent.

However, in general, two subgroups H and T of G may not be permutable in G but G maybe contain an element x such that $HT^x = T^xH$. Based on the observations, Guo, Shum and Skiba introduced the concept of conditionally permutable subgroup (in more general, the concept of X-permutable subgroup) [5]–[7]: let X be a non-empty subset of G. Then a subgroup A of G is said to be conditionally permutable (X-permutable) in G if for every subgroup T of G, there exists some $x \in G$ ($x \in X$ respectively) such that $AT^x = T^xA$. By using the conditionally permutable subgroups and X-permutable subgroups, authors have obtained some new elegant results on the structure of groups (cf. [5]–[8]).

By considering some local conditionally permutable subgroups, Huang and Guo [9] introduced the concept of s-conditionally permutable subgroup: a subgroup H of G is said to be s-conditionally permutable in G if, for every Sylow subgroup T of G, there exists some $x \in G$ such that $HT^x = T^xH$. By Sylow's theorem, we see that a subgroup H of G is s-conditionally permutable in G if and only if for every $p \in \pi(G)$, there exists a Sylow p-subgroup T such that $HT = TH$. As a development of s-conditionally permutable subgroups, Chen and Guo [10] introduced the concept of s-c-permutably embedded subgroups:

***Definition* 0.1** [10, Definition 1.1]. *A subgroup H of G is said to be s-conditionally permutably embedded (or in brevity, s-c-permutably embedded) in G if every Sylow subgroup of H is a Sylow subgroup of some s-conditionally permutable subgroup of G.*

Clearly, all permutable subgroups, s-permutable subgroups and s-conditionally permutable subgroups are s-c-permutably embedded. But the converse is not true in general (see, for example, Example 1-2 in [10]).

The purpose of this paper is to go further into the influence of s-c-permutably embedded subgroups on the structure of finite groups. Some new results are obtained and some known results are generalized.

1 Preliminary results

In this section, we give the related concepts and some basic results which are useful in the sequel.

***Lemma* 1.1** [10, Lemma 2.2]. *Suppose that G is a group, KG and $H \le G$. Then:*

(1) *If H is s-c-permutably embedded in G, then HK/K is s-c-permutably embedded in G/K.*

(2) *If $K \le H$ and H/K is s-c-permutably embedded in G/K, then H is s-c-permutably embedded in G.*

(3) *If HK/K is s-c-permutably embedded in G/K and $(|H|,|K|)=1$, then H is s-c-permutably embedded in G.*

(4) *If H is s-c-permutably embedded in G, then $H \cap K$ is s-c-permutably embedded in K.*

***Lemma* 1.2** [11, Lemma 3.1]. *Let N and L be normal subgroups in G such that P/L is a Sylow p-subgroup of NL/L and M/L is a maximal subgroup of P/L. If P_p is a Sylow p-subgroup of $P \cap N$, then P_p is a Sylow p-subgroup of N such that $D = M \cap N \cap P_p$ is a maximal subgroup of P_p and $M = LD$.*

***Lemma* 1.3** [12, Lemma 4.1]. *Let p be a prime dividing the order of G. Suppose that $(|G|, p-1)=1$ and the order of G is not divisible by p^3 and G is A_4-free. Then G is p-nilpotent.*

***Lemma* 1.4** [2, Theorem 1.8.17]. *Let N be a non-trivial normal subgroup of G. If $N \cap \Phi(G)=1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in $F(N)$.*

***Lemma* 1.5** [13, III, Lemma 3.3].

i) *If $N \trianglelefteq G$, $U \leqslant G$ and $N \leqslant \Phi(U)$, then $N \leqslant \Phi(G)$.*

ii) *If $M \trianglelefteq G$, then $\Phi(M) \le \Phi(G)$.*

Recall that, a class $\mathfrak{F}$ of groups is called a formation if it is closed under homomorphic image and subdirect product and every group G has a smallest normal subgroup (called $\mathfrak{F}$-residual) with quotient is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. A class of groups $\mathfrak{F}$ is said to be S-closed if every subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. We say a subgroup H of G is $\mathfrak{F}$-supplemented in G if G has a subgroup $T \in \mathfrak{F}$ such that $G = HT$. In this case, T is said to be an $\mathfrak{F}$-supplement of H in G. In particular, if $\mathfrak{F}$ is the class of all supersoluble groups (p-supersoluble groups), then an $\mathfrak{F}$-supplement is said to be a supersoluble supplement (a p-supersoluble supplement). We use $\mathfrak{U}$ to denote the formation of all supersoluble groups. The following Lemma is obvious.

***Lemma* 1.6.** *Let $\mathfrak{F}$ be a formation of groups. Suppose that a subgroup H of G has an $\mathfrak{F}$-supplement in G. Then:*

(1) *If $N \trianglelefteq G$, then HN/N has an $\mathfrak{F}$-supplement in G/N.*

(2) *If $H \le K \le G$ and $\mathfrak{F}$ is S-closed, then H has an $\mathfrak{F}$-supplement in K.*

***Lemma* 1.7** [14, Lemma 2.3]. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

***Lemma* 1.8** [15, Theorem 3.1]. *Let $\mathfrak{F}$ be a saturated formation contained $\mathfrak{U}$ and G has a soluble normal subgroup H such that $G/H \in \mathfrak{F}$. If for any maximal subgroup M of G, either $F(H) \le M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathfrak{F}$. The converse also holds, in the case where $\mathfrak{F} = \mathfrak{U}$.*

***Lemma* 1.9** [10, Theorem 3.2]. *Let G be a soluble group. If every maximal subgroup of every non-cyclic Sylow subgroup of G having no supersoluble supplement in G is s-c-permutably embedded in G, then G is supersoluble.*

Recall that a subgroup H of G is said to be a 2-maximal subgroup of G if H is a maximal subgroup of some maximal subgroup M of G.

2 Main results

***Theorem* 2.1.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of H having no supersoluble supplement in G is s-c-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is fasle and let (G,H) be a counterexample with $|G||H|$ is minimal. Then:

(1) $G/R \in \mathfrak{F}$, where R is an arbitrary minimal normal subgroup of G.

Obviously,

$(G/R)/(HR/R) \simeq G/HR \simeq (G/H)/(HR/H) \in \mathfrak{F}$

and $HR/R \simeq H/(H \cap R)$ is soluble. Let P/R be a non-cyclic Sylow p-subgroup of HR/R, where p is any prime divisor of $|HR/R|$, and M/R a maximal subgroup of P/R. If P_p is a Sylow p-subgroup of $P \cap H$, then by Lemma 1.2, P_p is a Sylow p-subgroup of H such that $L = M \cap H \cap P_p$ is a maximal subgroup of P_p and $M = LR$. Clearly, P_p is non-cyclic. By hypothesis, either L is s-c-permutably embedded in G or L has a supersoluble supplement in G. By Lemma 1.1 and Lemma 1.6, either $M/R = LR/R$ is s-c-permutably embedded in G or $M/R = LR/R$ has a supersoluble supplement in G. By the choice of G, $G/R \in \mathfrak{F}$.

(2) G has a unique minimal normal subgroup N, $G = [N]M$, where M is a maximal subgroup of G, and $N = O_p(G) = F(G) = C_G(N)$ for some prime p.

Since $\mathfrak{F}$ is a saturated formation, by (1), G has a unique minimal normal subgroup N and $\Phi(G) = 1$. Hence, there exists a maximal subgroup M of G such that $G = [N]M$. Since H is soluble, N is an elementary abelian p-group for some prime p. Clearly, $N \le O_p(G) \le F(G) \le C_G(N)$. Let $C = C_G(N)$. It is easy to see that $C \cap M \trianglelefteq G$. Hence $C = C \cap NM = N(C \cap M) = N$. Thus (2) holds.

(3) N is a non-cyclic Sylow p-subgroup of H.

By Lemma 1.1, Lemma 1.6 and Lemma 1.9, we know that H is supersoluble. By the choice of G, $H < G$. Let q be the largest prime divisor of $|H|$ and $Q \in Syl_q(H)$. Then $Q = O_q(H) \trianglelefteq G$. Since N is the unique minimal normal subgroup of G, $q = p$. Hence, by (2), we see that $N \subseteq Q = O_p(H) \subseteq O_p(G) = N$. By (1) and Lemma 1.7, we see that N is not cyclic. Thus (3) holds.

(4) Final contradiction.

Let G_p be a Sylow p-subgroup of G. Since $N \not\subseteq \Phi(G)$, $N \not\subseteq \Phi(G_p)$ by Lemma 1.5. So there exists a maximal subgroup P_1 of G_p such that $N \not\subseteq P_1$. Clearly, $N_1 = P_1 \cap N$ is a maximal subgroup of N. If N_1 has a supersoluble supplement in G, then there exists a supersoluble subgroup T of G such that $G = N_1T$. It is easy to see that $N \cap T \trianglelefteq NT = G.$. Hence $N \cap T = 1$ or $N \cap T = N$. If $N \cap T = N$, then $G = N_1T = T$ is supersoluble, a contradiction. If $N \cap T = 1$, then $N = N_1$, which is impossible. Hence we assume that N_1 is s-c-permutably embedded in G, that is, there exists an s-conditionally permutable subgroup A of G such that N_1 is a Sylow p-subgroup of A. In this case, for every $q \in \pi(G)$ and $q \ne p$, there exists a Sylow q-subgroup Q of G such that $AQ = QA$. Then $N_1 = N \cap P_1 = N \cap AQ \trianglelefteq AQ$ and consequently $Q \subseteq N_G(N_1)$. On the other hand, $N_1 = N \cap P_1 \trianglelefteq G_p$. Thus, $N_1 \trianglelefteq G$. It follows that $N \cap P_1 = 1$ and so $|N| = p$. Then by (1) and Lemma 1.7, we obtain that $G \in \mathfrak{F}$. This contradiction completes the proof.

***Theorem* 2.2.** *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ having no supersoluble supplement in G is s-c-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is fasle and let (G, H) be a counterexample with $|G||H|$ is minimal.

Let M be a maximal subgroup of G. If $F(H) \not\subseteq M$, then there exists a prime p dividing $|F(G)|$ such that $O_p(H) \not\subseteq M$. Thus $G = O_p(H)M$. It is clear that $\Phi(G) \cap F(H) = 1$. If not, we choose a minimal normal subgroup R of G contained in $\Phi(G) \cap F(H)$, then $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/R \in \mathfrak{F}$. Then, since $\mathfrak{F}$ is a saturated formation, we have that $G \in \mathfrak{F}$, a contradiction. By Lemma 1.5, $\Phi(O_p(H)) \subseteq \Phi(G) \cap F(H)$. Hence $\Phi(O_p(H)) = 1$. It follows from [2, Theorem 1.8.17] that $O_p(H)$ is an abelian p-group and consequently $O_p(H) \cap M \trianglelefteq G$. If $|O_p(H)| = p$, then $|F(H) : F(H) \cap M| = |G : M| = p$. Hence by Lemma 1.8, $G \in \mathfrak{F}$. This contradiction shows that $O_p(H)$ is a non-cyclic Sylow p-subgroup of $F(H)$. Let M_p be a Sylow p-subgroup of M. Then $G_p = O_p(H)M_p$ is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of G_p with $M_p \subseteq P_1$ and $P_2 = P_1 \cap O_p(H)$. Then $P_1 = P_1 \cap O_p(H)M_p = (P_1 \cap O_p(H))M_p = P_2M_p$ and $P_2 \cap M_p = O_p(H) \cap M_p$. Hence $|O_p(H) : P_2| = |O_p(H)M_p : P_2M_p| = |G_p : P_1| = p$, that is, P_2 is a maximal subgroup of $O_p(H)$. Since $O_p(H) \cap M \trianglelefteq G$, $P_2(O_p(H) \cap M)$ is a subgroup of $O_p(H)$. By the maximality of P_2 in $O_p(H)$, we know that $P_2(O_p(H) \cap M) = P_2$ or $P_2(O_p(H) \cap M) = O_p(H)$.

If $P_2(O_p(H)\cap M)=O_p(H)$, then $G=O_p(H)M=P_2M$. Since, obviously, $O_p(H)\cap M=P_2\cap M$, $O_p(H)=P_2$, a contradiction. Hence $P_2(O_p(H)\cap M)=P_2$. It follows that $O_p(H)\cap M\subseteq P_2$. Since $O_p(H)\cap M\trianglelefteq G$, $O_p(H)\cap M\subseteq (P_2)_G$. If $(P_2)_G\nleq M$, then $G=(P_2)_GM=$ $=P_2M$ and $O_p(H)=P_2(O_p(H)\cap M)=P_2$, a contradiction. Hence, $(P_2)_G\leq M$ and $(P_2)_G=O_p(H)\cap M$.

Suppose that P_2 has a supersoluble supplement N in G, then $G=P_2N=O_p(H)N$. If $O_p(H)\cap N\leq M$, then $O_p(H)\cap N\leq M\cap O_p(H)=(P_2)_G\leq P_2$. Therefore, $O_p(H)=P_2(O_p(H)\cap N)=P_2$, a contradiction. It follows that $O_p(H)\cap N\nleq M$.

Since $O_p(H)\cap N\trianglelefteq G$ and M is maximal in G, we have that $G=(O_p(H)\cap N)M$. By the modular law, $N=(O_p(H)\cap N)(M\cap N)$. It follows that $G=O_p(H)(M\cap N)$. By the modular law again, $M=(P_2)_G(M\cap N)$. Hence, $G=M(O_p(H)\cap N)=MN=$ $=(P_2)_GN$.

If $M\cap N$ is not maximal in N, then there exists a maximal subgroup N_1 of N such that $M\cap N<N_1$. Let $L=(P_2)_GN_1$. Since $(P_2)_G\leq M$, it follows that $(P_2)_G\cap N=(P_2)_G\cap(N\cap M)\leq(P_2)_G\cap N_1\leq$ $\leq(P_2)_G\cap N$. Hence, $(P_2)_G\cap N=(P_2)_G\cap N_1=$ $=(P_2)_G\cap(M\cap N)$. Since $G=(P_2)_GN$, $L=(P_2)_GN_1$, $M=(P_2)_G(M\cap N)$, we have that $M<L<G$, a contradiction. Therefore, $M\cap N$ is a maximal subgroup of N. Since N is supersoluble, it follows that $|F(H):F(H)\cap M|=|G:M|=|N:M\cap N|=p$, a prime. This implies that $F(H)\cap M$ is a maximal subgroup of $F(H)$. Then by Lemma 1.8, we obtain that $G\in\mathfrak{F}$, a contradiction.

Hence, by hypothesis, P_2 is s-c-permutably embedded in G. Then there exists an s-conditionally permutable subgroup A of G such that P_2 is a Sylow p-subgroup of A. Now, for every $q\in\pi(G)$ and $q\neq p$, there exists a Sylow q-subgroup Q of G such that $AQ\leq G$. Because $P_2=AQ\cap O_p(H)\trianglelefteq AQ$, we have that $Q\subseteq N_G(P_2)$. On the other hand, since $P_2=P_1\cap O_p(H)P_1$ and $O_p(H)$ is abelian,

$$G_p=O_p(H)M_p=O_p(H)P_1\subseteq N_G(P_2).$$

Thus, $P_2\trianglelefteq G$. This implies that $P_2=(P_2)_G\subseteq M$ and so $O_p(H)\cap M=P_2\cap M=P_2$. It follows that

$$|F(H):F(H)\cap M|=|G:M|=|O_p(H):O_p(H)\cap M|=p.$$

This indicates that $F(H)\cap M$ is a maximal subgroup of $F(H)$. By Lemma 1.8 again, we obtain that $G\in\mathfrak{F}$. The final contradiction completes the proof.

***Theorem* 2.3.** *A group G is p-supersoluble if and only if G has a normal p-soluble subgroup H such that G/H is p-supersoluble and every maximal subgroup of every Sylow p-subgroup of H having no p-supersoluble supplement in G is s-c-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is fasle and let (G,H) be a counterexample with $|G||H|$ is minimal. We proceed the proof via the following steps:

(1) If R is a minimal normal subgroup of G, then G/R is p-supersoluble.

Clearly, $(G/R)/(HR/R)\simeq G/HR\simeq(G/H)/(HR/H)$ is p-supersoluble and $HR/R\simeq H/(H\cap R)$ is p-soluble. Let P/R be a Sylow p-subgroup of HR/R and M/R a maximal subgroup of P/R. If P_p is a Sylow p-subgroup of $P\cap H$, then by Lemma 1.2, P_p is a Sylow p-subgroup of H such that $L=M\cap H\cap P_p$ is a maximal subgroup of P_p and $M=LR$. By hypothesis, either L is s-c-permutably embedded in G or L has a p-supersoluble supplement in G. By Lemma 1.1 and Lemma 1.6, we see that either $M/R=LR/R$ is s-c-permutably embedded in G or $M/R=LR/R$ has a p-supersoluble supplement in G. By the choice of (G,H), G/R is p-supersoluble.

(2) $O_{p'}(G)=1$ and G has a unique minimal normal subgroup N such that $N=C_G(N)=O_p(G)\Phi(G)$ and $|N|\neq p$.

In fact, if $O_{p'}(G)\neq 1$, then, by (1), $G/O_{p'}(G)$ is p-supersoluble. It follows that G is p-supersoluble, a contradiction. Hence, $O_{p'}(G)=1$. Since the class of all p-supersoluble groups is a saturated formation, G has a unique minimal normal subgroup N and $N\nsubseteq\Phi(G)$. Obviously, $N=C_G(N)=O_p(G)$. By (1) and Lemma 1.7, $|N|\neq p$.

(3) If $H\leq D\trianglelefteq G$ and $D<G$, then D is p-supersoluble.

It is clear that D/H is p-supersoluble and (D,H) satisfies the hypothesis by Lemma 1.1 (4) and Lemma 1.6. Hence, by the choice of (G,H), D is p-supersoluble.

(4) Let H_p be a Sylow p-subgroup of H. Then $1\neq H_p\neq N$ and so H_p is not normal in G.

By hypothesis, obviously, $H_p \neq 1$. If $H_p = N$, then, by (2), $|H_p| > p$. Since $H_p \not\subseteq \Phi(G)$ and $H_p \trianglelefteq G$, $H_p \not\subseteq \Phi(G_p)$ by Lemma 1.5, where G_p is a Sylow p-subgroup of G. Hence, there exists a maximal subgroup P_1 of G_p such that $H_p \not\subseteq P_1$. Let $E = H_p \cap P_1$. Then E is a maximal subgroup of H_p. If E has a p-supersoluble supplement T in G, then $|G:T| \leq |E|$. Since $H_pT = ET = G$ and H_p is an abelian minimal normal subgroup of G, $G = [H_p]T$. This implies that $|G:T| = |H_p| > |E|$, a contradiction. Hence E is s-c-permutably embedded in G, that is, there exists an s-conditionally permutable subgroup A of G such that E is a Sylow p-subgroup of A. So for every $q \in \pi(G)$ and $q \neq p$, there exists a Sylow q-subgroup Q of G such that $QA = AQ$. Thus $E = H_p \cap P_1 = H_p \cap AQ \trianglelefteq AQ$. It follows that $Q \leq N_G(E)$. Besides, $E = H_p \cap P_1 \trianglelefteq G_p$. Therefore $E \trianglelefteq G$. This induces that $E = 1$ and so $|H_p| = p$, a contradiction. Thus (4) holds.

(5) $G = [N]M$, where M is a p-supersoluble maximal subgroup of G such that $p \,||\, |M|$ and $O_p(M) = 1$.

By (1) and (2), G has a p-supersoluble maximal subgroup M such that $G = [N]M$. By [2, Lemma 1.7.11], $O_p(G/C_G(N)) = O_p(G/N) = 1$. Hence $O_p(M) = 1$. Assume that $p \nmid |M|$. Then p does not divide $|G/N|$. Since $N \subseteq H$, H/N is a p'-group, which contradicts (4).

(6) $H = G$.

Assume that $H \neq G$. Consider the subgroup $H \cap M$. Since $H = H \cap NM = N(H \cap M)$ and $N \neq H$, $H \cap M \neq 1$. By (2) and (3), H is p-supersoluble and $O_{p'}(H) = 1$. It follows from [11, Lemma 3.3] that H is supersoluble. This implies that p is the largest prime divisor of $|H|$ and so the Sylow p-subgroup P of $H \cap M$ is normal in $H \cap M$. Hence P char $H \cap M \trianglelefteq M$. Since $O_p(M) = 1$, $P = 1$. It follows that N is a Sylow p-subgroup of H, which contradicts (4).

(7) Every maximal subgroup of every Sylow p-subgroup of G has a p-supersoluble supplement in G.

Let G_p be a Sylow p-subgroup of G and P_1 a maximal subgroup of G_p. If $N \subseteq P_1$, then, by (5), P_1 has a p-supersoluble supplement M in G. Assume that $N \not\subseteq P_1$ and P_1 is s-c-permutably embedded in G. Then there exists an s-conditionally permutable subgroup A of G such that P_1 be a Sylow p-subgroup of A. By the same discussion as in (4), we obtain that $P_1 \trianglelefteq G$ and consequently $N \subseteq P_1$, a contradiction.

(8) Final contradiction.

By (7) and [11, Theorem 3.4], we obtain that G is p-supersoluble. This final contradiction completes the proof.

***Theorem* 2.4.** *Let p be the smallest prime dividing the order of a p-soluble group G and P a Sylow p-subgroup of G. If every 2-maximal subgroup of P is s-c-permutably embedded in G and G is A_4-free, then G is p-nilpotent.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. We proceed with our proof as follows:

(1) G/N is p-nilpotent, for every non-trivial normal subgroup N of G.

If some Sylow p-subgroup of G is contained in N, then, obviously, G/N is p-nilpotent. Hence, we may assume that N does not contain any Sylow p-subgroup of G. Let PN/N be a Sylow p-subgroup of G/N, where P is a Sylow p-subgroup of G, and M_2/N a 2-maximal subgroup of PN/N. It is easy to see that $M_2 = PN \cap M_2 = (P \cap M_2)N$. Let $P_2 = P \cap M_2$. Since $P \cap M_2 \cap N = P \cap N$, $p^2 = |PN/N : M_2/N| = |PN : (P \cap M_2)N| = |P : P_2|$. Hence P_2 is a 2-maximal subgroup of P and $M_2 = P_2N$. By Lemma 1.1, $M_2/N = P_2N/N$ is s-c-permutably embedded in G/N. This shows that G/N satisfies the hypothesis. The minimal choice of G implies that G/N is p-nilpotent.

(2) G has a unique minimal normal subgroup $H = C_G(H)$ and $\Phi(G) = 1$.

Since the class of all p-nilpotent groups is a saturated formation, G has a unique minimal normal subgroup, say H, and $\Phi(G) = 1$. Because G is a p-soluble group, H is a p-group or a p'-group. If H is a p'-group, then G is p-nilpotent. Hence H is an elementary abelian p-group. Now, by the similar argument as in the proof (2) of Theorem 2.1, we can know that $H = C_G(H)$.

(3) $|H| \geq p^2$.

If $|H| = p$, then $G/H = G/C_G(H) \lesssim Aut(H)$ is a cyclic group of order $p-1$. Since p is the smallest prime of $|G|$, $G = C_G(H)$, that is, $H \subseteq Z(G)$. This induces that G is p-nilpotent, a contradiction. Thus (3) holds.

(4) Final contradiction.

By (2), we see that there exists a maximal subgroup M of G such that $G=[H]M$. Let M_p be a Sylow p-subgroup of M. Then $G_p = M_pH$ is a Sylow p-subgroup of G. By Lemma 1.3, we see that $|G_p| \geq p^3$. Let G_0 be a 2-maximal subgroup of G_p with $M_p \subseteq G_0$ and $H_1 = G_0 \cap H$. Then $|H:H_1|=|H:G_0\cap H|=|HG_0:G_0|=|G_p:G_0|=p^2$. Hence H_1 is a 2-maximal subgroup of H. By hypothesis, G_0 is s-c-permutably embedded in G. Hence there exists an *s*-conditionally permutable subgroup A of G such that G_0 is a Sylow p-subgroup of A. Let q be an arbitrary prime divisor of $|G|$ with $q \neq p$. Since A is s-conditionally permutable in G, there exists a Sylow q-subgroup Q of G such that $AQ=QA$. As H_1 is a 2-maximal subgroup of H and $H_1=G_0\cap H\subseteq AQ\cap H\subseteq H$, we have that $H_1 = AQ\cap H$ or $AQ\cap H = H$ or $H_1 \subset AQ\cap H \subset H$. If $AQ \cap H = H$, then $H\subseteq AQ$ and so $G_p = M_pH \subseteq AQ$, which is clearly impossible. If $H_1 \subset AQ\cap H\subset H$, then $AQ\cap H$ is a maximal subgroup of H. Let $H_2 = AQ\cap H$. Since $H_2 = AQ\cap H \trianglelefteq AQ$ and $H_2 \trianglelefteq H$, , $AQ\subseteq N_G(H_2)$ and $G_p = G_0H \leq AH \leq N_G(H_2)$. This implies that $H_2 \trianglelefteq G$. However, because H is the minimal normal subgroup of G, we have that $H_2=1$. It follows that $|H|=p$, a contradiction. Hence $H_1 = AQ\cap H \trianglelefteq AQ$. It follows that $AQ\subseteq N_G(H_1)$. On the other hand, since $H_1 = G_0\cap H \trianglelefteq G_0$ and H is an abelian group, $G_p = G_0H \subseteq N_G(H_1)$. This shows that $H_1 \trianglelefteq G$. Consequently, $H_1 = 1$ and so $|H|=p^2$. It follows that $|Aut(H)|=(p+1)p(p-1)^2$. Since $q>p$ and $G/H = G/C_G(H) \lesssim Aut(H)$, $q=p+1$. This induces that $p=2$, $q=3$. Let x be an element of order 3. Thus $[H]\langle x\rangle$ is a subgroup of G, which contradicts the fact that G is A_4-free. The final contradiction completes the proof.

***Remark* 2.4.1.** In Theorem 2.4, we cannot omit the assumption that G is A_4-free in general. For example, $G=A_4$. It is clear that every 2-maximal subgroup of the Sylow 2-subgroups of G is the identity subgroup and of course, is s-c-permutably embedded in G. But G is not 2-nilpotent.

***Corollary* 2.4.1.** *Let G be a soluble group. Suppose that for each prime divisor p of $|G|$ and $P\in Syl_p(G)$, every 2-maximal subgroup of P is s-c-permutably embedded in G and G is A_4-free, then G is a Sylow tower group* (see [2, p. 49]).

***Theorem* 2.5.** *Let G be a group and N a soluble normal subgroup of G such that G/N is a Sylow tower group. If, for every prime p dividing the order of N and $P\in Syl_p(N)$, every 2-maximal subgroup of P is s-c-permutably embedded in G and G is A_4-free, then G is a Sylow tower group.*

Proof. By Lemma 1.1 (4) and Corollary 2.4.1, we can see that N is a Sylow tower group by induction. Let r be the largest prime number in $\pi(N)$ and $R\in Syl_r(N)$. Then R char $N\trianglelefteq G$ and so $R\trianglelefteq G$. By Lemma 1.1 (1) and induction, G/R is a Sylow tower group. Let q be the largest prime divisor of $|G|$ and Q a Sylow q-subgroup of G. Then $RQ/R \trianglelefteq G/R$ and thereby $RQ\trianglelefteq G$. If $q=r$, then, obviously, G is a Sylow tower group by induction. Hence, we assume that $r<q$.

Case 1. $RQ<G$. In this case, RQ is a Sylow tower group by Theorem 2.4 and induction. It follows that $Q\trianglelefteq RQ$ and so $Q\trianglelefteq G$. Thus G is a Sylow tower group.

Case 2. $G=RQ$. Let L be a minimal normal subgroup of G with $L\subseteq R$. Then the quotient group G/L (with respect to N/L) satisfies the hypothesis. Hence, by induction, G/L is a Sylow tower group. Since the class of all Sylow tower groups is a saturated formation, $L \not\subseteq \Phi(G)$ and L is the unique minimal normal subgroup of G which is contained in R. Therefore, $L=F(R)=R$ by Lemma 1.4. In particular, R is an elementary abelian group.

If R is a cyclic subgroup of order r, then $r<q$ implies that G is r-nilpotent by [16, (10.1.9)] and so $G=R\times Q$. Hence G is a Sylow tower group. Now assume that $|R|\geq r^2$. Let R_1 be a 2-maximal subgroup of R. By hypothesis, R_1 is s-c-permutably embedded in G. Hence there exists an s-conditionally permutable subgroup A of G such that R_1 is a Sylow r-subgroup of A. Then, for some $Q_1\in Syl_q(G)$, we have $AQ_1\leq G$. Since $R_1 = R\cap AQ_1 \trianglelefteq AQ_1$, $AQ_1\subseteq N_G(R_1)$. This implies that $R_1\trianglelefteq G$. But, because R is the minimal normal subgroup of G, we have that $R_1=1$ and so $|R|=r^2$. Since $Q\subseteq Aut(R)$ and $|Aut(R)|=(r+1)r(r-1)^2$, $q=3$ and $r=2$, which contradicts the fact that G is A_4-free. The proof is completed.

3 Some applications of the results

Theorems 2.1–2.3 have many corollaries. We state only some special cases of theorem which can be found in the literature.

Theorem 2.1 immediately implies

***Corollary* 3.1** (Huang, Guo [9]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of H is s-conditionally permutable in G.*

***Corollary* 3.2** (Chen, Guo [10]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup H such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of H is s-c-permutably embedded in G.*

Recall that, let X be a non-empty subset of G. Then a subgroup H of G is c-semipermutable (X-semipermutable) in G if there is a minimal supplement T of H in G such that H is T-per-mutable (X-permutable) with all subgroups of T (see [8], [17]). Clearly, if a subgroup H of G of prime power order is c-semipermutable (X-semipermutable) in G, then H is s-conditionally permutable in G and consequently is s-c-permutably embedded in G. Hence we immediately have the following corollary.

***Corollary* 3.3** (Hu, Guo [17]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of H is c-semipermutable in G.*

From Theorem 2.3, we have

***Corollary* 3.4** (Zha, Guo, Li [18]). *Let G be a p-soluble group. Then G is p-supersoluble if and only if G has a normal subgroup N such that G/N is p-supersoluble and every maximal subgroup of every Sylow p-subgroup of N having no p-supersoluble supplement in G is s-conditionally permutable in G.*

From Theorem 2.2, we obtain

***Corollary* 3.5** (Ramadan [19]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(G)$ are normal in G, then G is supersoluble.*

***Corollary* 3.6** (Ramadan [19]). *Let G be a soluble group, and E a normal subgroup of G such that G/E is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

***Corollary* 3.7** (Asaad, Ramadan, Shaalan [20]). *Suppose that G/H is supersoluble. If H is supersoluble and all maximal subgroups of any Sylow subgroup of $F(H)$ are s-permutable in G, then G is supersoluble.*

***Corollary* 3.8** (Asaad [21]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a soluble group with a normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are s-permutable in G, then $G \in \mathfrak{F}$.*

***Corollary* 3.9** (Huang, Guo [9]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of $F(H)$ is s-conditionally permutable in G.*

***Corollary* 3.10** (Chen, Guo [10]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of Sylow subgroups $F(H)$ is s-c-permutably embedded in G.*

***Corollary* 3.11** (Hu, Guo [17]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every maximal subgroup of Sylow subgroups $F(H)$ is c-semipermutable in G.*

***Corollary* 3.12** (Chen, Li [22]). *A group G is supersoluble if and only if there exists a soluble normal subgroup H of G such that G/H is supersoluble and every maximal subgroup of every Sylow subgroup of the Fitting subgroup $F(H)$ of H is $F(H)$-semipermutable in G.*

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